

CHAPTER 16 GENERAL RELATIVISTIC TERMS FOR PROPAGATION, TIME, AND COORDINATES

VLBI TIME DELAY

There are many papers dealing with relativistic effects which must be accounted for in VLBI processing; see Robertson (1975), Finkelstein, et al. (1983), Hellings (1986), Pavlov (1985), Cannon, et al. (1986), Soffel, et al. (1986), Zeller, et al. (1986), Zhu and Groten (1988). As pointed out by Boucher (1986), the relativistic correction models proposed in various articles are not quite compatible. Nearly all the models in these papers were derived in the barycentric or B-frame. The following material is taken largely from the article of Zhu and Groten (1988).

Event one is the arrival of a radio signal at antenna 1; event two is the same wave front at antenna 2. The delay expressed in coordinate time is,

$$\Delta t_{\text{obs}} = -[\vec{R}_1(t_1) - \vec{R}_2(t_2)] \cdot \hat{K}/c + \Delta t_{\text{grav}}, \quad (1)$$

[see Finkelstein, et al. (1983), Zeller, et al. (1986)]. Here \hat{K} is the direction of the source seen from the barycenter. Δt_{grav} is called the gravitational time delay. \vec{R}_1 and \vec{R}_2 are barycenter vectors of antenna 1 and 2, respectively. In the derivation, it is assumed that $|\vec{R}_0| \gg |\vec{R}_1|$, where $|\vec{R}_0|$ is the distance of the solar system barycenter from the source. For radio stars in our galaxy, relevant corrections must be added.

Defining the baseline in the B-frame as,

$$\vec{b} = \vec{R}_1(t_1) - \vec{R}_2(t_1), \quad (2)$$

it is easy to obtain (Robertson, 1975),

$$\Delta t_{\text{obs}} = \Delta t_0 \{1 + (\dot{\vec{R}}_2 \cdot \hat{K})/c + (\ddot{\vec{R}}_2 \cdot \hat{K})^2/c^2 + (\ddot{\vec{R}}_2 \cdot \hat{K}) \Delta t_0/2c\} + \Delta t_{\text{grav}}, \quad (3)$$

where $\Delta t_0 = -(\vec{b} \cdot \hat{K})/c$.

The barycentric position vector of antenna i can also be written as $\vec{R}_i = \vec{R} + \vec{r}_i$, where \vec{R} is the barycentric position of the geocenter, and \vec{r}_i is the geocentric position vector of the same antenna measured in the B-frame. Then equation (2) is equivalent to

$$\vec{b} = \vec{r}_1(t_1) - \vec{r}_2(t_1). \quad (2')$$

Equation (3) is the delay expressed in coordinate time. It must be transformed to measurable atomic time. The baseline \vec{b} in the above equations has to be changed into its counterpart expressed in the geocentric or E-frame. Define this baseline in the E-frame as $\vec{B} = \vec{r}_1(\tau_1) - \vec{r}_2(\tau_1)$.

Zhu and Groten (1988) use the approach similar to the one used in Zeller, et al. (1986) in order to account for these coordinate transformations. The coordinates in the E-frame are expressed by $(c\tau, \hat{x})$. The coordinate transformations between these two systems take the form (Fukushima, et al. 1986)

$$\Delta t = \{1 + \langle \phi + v^2/2 \rangle_p / c^2\} \Delta \tau + (\vec{v} \cdot \Delta \hat{x}) / c^2, \quad (4-a)$$

$$\Delta \vec{x} = \{1 - (\phi + \langle \phi + v^2/2 \rangle) / c^2\} \Delta \hat{x} + \vec{v} \Delta \tau + (\vec{v} \cdot \Delta \hat{x}) \vec{v} / 2c^2, \quad (4-b)$$

where ϕ and \vec{v} are Newtonian potential and velocity at the center of mass of the Earth, respectively, $\langle \rangle$ denotes the long time average, $\langle \rangle_p$ means the periodic terms with the long time average being removed.

If $\dot{\vec{r}}_2 = \vec{v}_2$, the final $\Delta \tau_{\text{obs}}$ derived in the B-frame is

$$\begin{aligned} \Delta \tau_{\text{obs}}^{\text{B}} &= \Delta \tau_0 [1 + (\vec{v} + \vec{v}_2) \cdot \hat{K} / c + (\vec{v} \cdot \hat{K})^2 / c^2 + 2(\vec{v} \cdot \hat{K})(\vec{v}_2 \cdot \hat{K}) / c^2 - \phi / c^2 \\ &\quad - (\phi + v^2/2) / c^2 - (\vec{v} \cdot \vec{v}_2) / c^2] + (\vec{B} \cdot \vec{v}) / c^2 + (\vec{v} \cdot \vec{v})(\hat{K} \cdot \vec{v}) / 2c^3 \\ &\quad + (\vec{B} \cdot \vec{v})(\vec{v}_2 \cdot \hat{K}) / c^3 + \Delta \tau_{\text{grav}}, \end{aligned} \quad (5)$$

with

$$\Delta \tau_0 = - (\vec{B} \cdot \hat{K}) / c. \quad (6)$$

This result differs from that given in Zeller, et al. (1986) only by an additional term $-\Delta \tau \phi / c^2$ which is the consequence of the general relativistic effect on the spatial coordinate transformation. This term amounts to 10^{-8} (in which the secular part is at 10^{-8} , and the periodic part is at 10^{-10}). Therefore it

should not be neglected.

As derived in Finkelstein, et al. (1983), Zeller, et al. (1986) etc. the gravitational time delay is

$$\Delta\tau_{\text{grav}} = 2 \sum_j (Gm_j/c^3) \ln [(|\vec{R}_{1j}| - \vec{R}_{1j} \cdot \hat{K}) / (|\vec{R}_{2j}| - \vec{R}_{2j} \cdot \hat{K})], \quad (7)$$

in which

$$\vec{R}_{ij} = \vec{R}_i - \vec{R}_j$$

$$\left[\begin{array}{ll} i = 1, 2, & j = 1 \dots n \\ i = \text{antenna}, & j = \text{attracting body} \end{array} \right]$$

is the vector from body j to antenna i . If the attracting body is the Earth, equation (7) takes the form,

$$\Delta\tau_{\text{grav}}(E) = \frac{2GM_{\oplus}}{c^3} \ln \left[\frac{1 + \sin E_1}{1 + \sin E_2} \right], \quad (8)$$

in which E_i is the source elevation angle viewed at antenna i . In any other cases, since $|\vec{R}_{ij}| \gg |\vec{b}|$ equation (7) reduces to

$$\Delta\tau_{\text{grav}}(j) \approx 2 \frac{Gm_j}{c^3 |\vec{R}_{oj}|} \frac{-\vec{b} \cdot \hat{K} + \vec{b} \cdot \hat{n}_{oj}}{1 - \hat{n}_{oj} \cdot \hat{K}} \quad (9)$$

where \vec{R}_{oj} is the vector from body j to geocenter, and

$$\hat{n}_{oj} = \vec{R}_{oj} / |\vec{R}_{oj}|.$$

As pointed by Zeller, et al. (1986), when \hat{n}_{oj} and \hat{K} are nearly parallel, that is, the source is near the limb of the body, equation (9) will not be accurate enough. In any other case, equation (9) could be used instead of equation (7).

Another expression used for the VLBI delay observable (Shapiro, et al., 1989) is

$$\begin{aligned} \Delta\tau = & - (\hat{K} \cdot \vec{b}) / c \\ & - [(\vec{b} \cdot \vec{v}) - (\hat{K} \cdot \vec{b})(\hat{K} \cdot \vec{v})] / c^2 \\ & + \Delta\tau^{\text{pm}} \end{aligned}$$

$$\begin{aligned}
& + (\hat{K} \cdot \vec{b}) (\hat{K} \cdot \vec{r}_2) / c^2 \\
& + [(\vec{b} \cdot \vec{v}) (\hat{K} \cdot \vec{v}) / 2 + (\hat{K} \cdot \vec{b}) (\vec{v} \cdot \vec{v}) / 2 - (\hat{K} \cdot \vec{b}) (\hat{K} \cdot \vec{v})^2] / c^3 \\
& + 2U(\hat{K} \cdot \vec{b}) / c + (\vec{b} \cdot \vec{v}) (\hat{K} \cdot \vec{r}_2) / c^3,
\end{aligned}$$

where $\Delta\tau$ is the delay defined to be $\tau_2 - \tau_1$ with τ_1 being the atomic time reading at site 1 when a wavefront arrives from a distant source, and τ_2 is the atomic time at site 2 when this same wavefront arrives there, \hat{K} is the unit vector in the direction of the source as viewed from the solar system barycenter in the J2000.0 coordinate system; \vec{b} is the baseline vector $\vec{r}_2 - \vec{r}_1$, at atomic time τ_1 ; \vec{r}_1 and \vec{r}_2 are the vectors from the center of the Earth to site 1 and site 2, respectively; \vec{v} is the velocity of the center of the Earth in solar system barycentric (SSB) coordinates at the SSB coordinate time corresponding to τ_1 ; \vec{r}_2 is geocentric velocity of site 2 at τ_1 ; $\Delta\tau^{\text{pm}}$ is the propagation medium delay and light deflection defined below; U is the gravitational potential at site 1 computed at the SSB coordinate time corresponding to τ_1 and is given by:

$$U = \sum_k \frac{GM_k}{c^2 r_k},$$

where G is the gravitational constant, M_k is the mass of the k^{th} body, and r_k is the distance from the k^{th} body to site 1. The sum k includes all bodies except the Earth itself. In practice, only the Sun need be included.

The $\Delta\tau^{\text{pm}}$ term is given by

$$\begin{aligned}
\Delta\tau^{\text{pm}} = & \frac{(1+\gamma)r_o}{c} \ln \left[\frac{r_o(1+\hat{r}_o \cdot \hat{K}) + \Delta\vec{r}_1 \cdot (\hat{r}_o + \hat{K})}{r_o(1+r_o \cdot \hat{K}) + \Delta\vec{r}_2 \cdot (r_o + \hat{K})} \right] \\
& + \frac{(1+\gamma)r_w}{c} \ln \left[\frac{r_1(1+\hat{r}_1 \cdot \hat{K}/r_1)}{r_2(1+\hat{r}_2 \cdot \hat{K}/r_2)} \right] \\
& + \tau^a + \tau^i,
\end{aligned}$$

where

$$r_o = GM_\odot/c^2 \text{ and } r_w = GM_\oplus/c^2,$$

and M_\odot and M_\oplus are the masses of the Sun and the Earth, respectively; \hat{K} is the unit vector in the direction from Sun to the radio source; \hat{r}_o is the unit vector in the direction from the Sun to the Earth; r_o is the distance from the Sun to the Earth; r_1 and r_2 are the magnitudes of these vectors, and τ^a and τ^i are the differential delays due to the propagation of the signals through the atmosphere and the ionosphere, respectively. The variable γ is the γ -factor in the parametrized post-Newtonian gravitational theory. Note: the above expressions are only valid for sources which are an infinite distance from the Earth, and, in particular, are not valid for

signals from GPS satellites.

PROPAGATION CORRECTION FOR LASER RANGING

The space-time curvature near a massive body requires a correction to the Euclidean computation of range, ρ . This correction in seconds, Δt , is given by (Holdridge, 1967)

$$\Delta t = \frac{(1 + \gamma)GM}{c^3} \ln \left[\frac{R_1 + R_2 + \rho}{R_1 + R_2 - \rho} \right], \quad (10)$$

where

c = speed of light,

γ = PPN parameter equal to 1 in General Relativity,

R_1 = distance from the body's center to the beginning of the light path,

R_2 = distance from the body's center to the end of the light path,

GM = gravitational parameter of the deflecting body.

For near-Earth satellites, working in the geocentric frame of reference, the only body to be considered is the Earth (Ries, Huang, and Watkins, 1988). For lunar laser ranging, which is formulated in the solar system barycentric reference frame, the Sun and the Earth must be considered (Newhall, Williams, and Dickey, 1987).

In the computation of the instantaneous space-fixed positions of a station and a lunar reflector in the analysis of LLR data, the body-centered coordinates of the two sites are affected by a scale reduction and a Lorentz contraction effect (Martin, Torrence, and Misner, 1985). The scale effect is about 15 cm in the height of a tracking station, while the maximum value of the Lorentz effect is about 3 cm. The equation for the transformation of \vec{r} , the geocentric position vector of a station expressed in the geocentric frame, is

$$\vec{r}_b = \vec{r} \left[1 - \frac{\gamma\Phi}{c^2} - L \right] - \frac{1}{2} \left(\frac{\vec{V} \cdot \vec{r}}{c^2} \right) \vec{V}, \quad (11)$$

where

\vec{r}_b = station position expressed in the barycentric frame,

Φ = gravitational potential at the geocenter (excluding the Earth's mass),

\vec{V} = barycentric velocity of the Earth,

$L = \langle (\Phi + V^2/2)/c^2 \rangle = 1.4808 \times 10^{-8}$, where $\langle \rangle$ denotes the long time average.

A similar equation applies to the selenocentric reflector coordinates; the maximum value of the Lorentz effect is about 1 cm (Newhall, Williams, and Dickey, 1987).

EPOCH AND TIME INTERVALS

The transformation from solar system barycentric coordinate time t (i. e. TDB) to atomic clock time, τ (i. e. TAI), involves a constant offset, an annual and diurnal term, and other small periodic terms. This transformation has usually been given in the past in terms of the Earth's orbital elements, terrestrial coordinates, and UT. With the near universal availability of ephemeris tapes, a more convenient expression is given with sufficient accuracy by Moyer (1981)

$$\begin{aligned}
 t - \tau = & \Delta T_A + \frac{2}{c^2} (\dot{\vec{r}}_B^S \cdot \vec{r}_B^S) + \frac{1}{c^2} (\dot{\vec{r}}_B^C \cdot \vec{r}_E^B) \\
 & + \frac{1}{c^2} (\dot{\vec{r}}_E^C \cdot \vec{r}_A^E) + \frac{\mu_J}{c^2(\mu_S + \mu_J)} (\dot{\vec{r}}_J^S \cdot \vec{r}_J^S) \\
 & + \frac{\mu_{SA}}{c^2(\mu_S + \mu_{SA})} (\dot{\vec{r}}_{SA}^S \cdot \vec{r}_{SA}^S) + \frac{1}{c^2} (\dot{\vec{r}}_S^C \cdot \vec{r}_B^S), \quad (12)
 \end{aligned}$$

where $\Delta T_A = 32^s.184$. The quantities \vec{r}_x^y and $\dot{\vec{r}}_x^y$ are the position and velocity of point x relative to point y . The superscripts and subscripts are defined as follows:

A = location of station clock on Earth which reads International Atomic Time (TAI),

E = Earth,

B = Earth-Moon barycenter,

M = Moon,

S = Sun,
C = solar system barycenter,
J = Jupiter,
SA = Saturn.

The quantity μ_x is the gravitational coefficient of body X. An expression with higher accuracy is available in Fairhead, et al. (1988). No relativistic time coorections are required for satellite laser ranging in the geocentric frame (i. e. TDT \equiv TAI + ΔT_A).

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