

## CHAPTER 7 SOLID EARTH TIDES

The solid Earth tide model is based on an abbreviated form of the Wahr model (Wahr, 1981) using the Earth model 1066A of Gilbert and Dziewonski (1975).

The Love numbers for the induced free space potential,  $k$ , and for the vertical and horizontal displacements,  $h$  and  $\ell$ , have been taken from Wahr's thesis, Tables 13 and 16. The long period, diurnal, and semi-diurnal terms are included. Third degree terms are neglected.

### Calculation of the Potential Coefficients

The solid tide induced free space potential is most easily modelled as variations in the standard geopotential coefficients  $C_{nm}$  and  $S_{nm}$  (Eanes, et al., 1983). The Wahr model (or any other having frequency dependent Love numbers) is most efficiently computed in two steps. The first step uses a frequency independent Love number  $k_2$  and an evaluation of the tidal potential in the time domain from a lunar and solar ephemeris. The second step corrects those arguments of a harmonic expansion of the tide generating potential for which the error from using the  $k_2$  of Step 1 is above some cutoff.

The changes in normalized second degree geopotential coefficients for Step 1 are:

$$\Delta \bar{C}_{20} = \frac{1}{\sqrt{5}} k_2 \frac{R_e^3}{GM_\oplus} \sum_{j=2}^3 \frac{GM_j}{r_j^3} P_{20}(\sin\phi_j), \quad (1a)$$

$$\Delta \bar{C}_{21} - i\Delta \bar{S}_{21} = \frac{1}{3} \sqrt{\frac{3}{5}} k_2 \frac{R_e^3}{GM_\oplus} \sum_{j=2}^3 \frac{GM_j}{r_j^3} P_{21}(\sin\phi_j) e^{-i\lambda_j}, \quad (1b)$$

$$\Delta \bar{C}_{22} - i\Delta \bar{S}_{22} = \frac{1}{12} \sqrt{\frac{12}{5}} k_2 \frac{R_e^3}{GM_\oplus} \sum_{j=2}^3 \frac{GM_j}{r_j^3} P_{22}(\sin\phi_j) e^{-i2\lambda_j}, \quad (1c)$$

where

$k_2$  = nominal second degree Love number,

$R_e$  = equatorial radius of the Earth,

$GM_{\oplus}$  = gravitational parameter for the Earth,

$GM_j$  = gravitational parameter for the Moon ( $j=2$ ) and Sun ( $j=3$ ),

$r_j$  = distance from geocenter to Moon or Sun,

$\phi_j$  = body fixed geocentric latitude of Moon or Sun,

$\lambda_j$  = body fixed east longitude (from Greenwich) of Sun or Moon.

The changes in normalized coefficients from Step 2 are:

$$\Delta \bar{C}_{nm} - i \Delta \bar{S}_{nm} = A_m \sum_{s(n,m)} \delta k_s H_s \begin{pmatrix} 1 \\ -i \end{pmatrix} \begin{matrix} n+m \text{ even} \\ n+m \text{ odd} \end{matrix} e^{i\theta_s}, \quad (2)$$

where

$$A_m = \frac{(-1)^m}{R_e \sqrt{4\pi} (2 - \delta_{om})}, \quad \delta_{om} = \begin{cases} 1 & m=0 \\ 0 & m \neq 0 \end{cases}$$

$\delta k_s$  = difference between Wahr model for  $k$  at frequency  $s$  and the nominal value  $k_2$  in the sense  $k_s - k_2$ ,

$H_s$  = amplitude (m) of term at frequency  $s$  from the Cartwright and Tayler (1971) and Cartwright and Edden (1973) harmonic expansion of the tide generating potential,

$$\theta_s = \bar{n} \cdot \bar{\beta} = \sum_{i=1}^6 n_i \beta_i,$$

where

$\bar{n}$  = six vector of multipliers of the Doodson variables,

$\bar{\beta}$  = the Doodson variables, and

$$\delta S_{20} = 0.$$

The Doodson variables are related to the fundamental arguments of the nutation series (see Chapter 5) by:

$$s = F + \Omega = \beta_2 \text{ (Moon's mean longitude)}, \quad (3)$$

$$h = s - D = \beta_3 \text{ (Sun's mean longitude)},$$

$p = s - \ell = \beta_4$  (Longitude of Moon's mean perigee),

$N' = -\Omega = \beta_5$  (Negative longitude of Moon's mean node),

$p_1 = s - D - \ell' = \beta_6$  (Longitude of Sun's mean perigee), and

$\tau = \theta_g + \pi - s = \beta_1$  (Time angle in lunar days reckoned from lower transit), where

$\theta_g$  = mean sidereal time of the conventional zero meridian.

The normalized geopotential coefficients ( $\bar{C}_{nm}$ ,  $\bar{S}_{nm}$ ) are related to the unnormalized coefficients ( $C_{nm}$ ,  $S_{nm}$ ) by

$$C_{nm} = N_{nm} \bar{C}_{nm},$$

$$S_{nm} = N_{nm} \bar{S}_{nm},$$

$$N_{nm} = \sqrt{\frac{(n-m)! (2n+1) (2-\delta_{om})}{(n+m)!}}$$

Using a nominal  $k_2$  of 0.30 and an amplitude cutoff of  $9 \times 10^{-12}$  change in normalized geopotential coefficients, the summation  $S(n,m)$  requires six terms for the diurnal species ( $n=2, m=1$ ) modifying  $\bar{C}_{21}$  and  $\bar{S}_{21}$  and two semi-diurnal terms ( $n=2, m=2$ ) modifying  $\bar{C}_{22}$  and  $\bar{S}_{22}$ . With the exception of the zero frequency tide, no long period terms are necessary. Table 7.1 gives required quantities for correcting the (2,1) and (2,2) coefficients. The correction to  $C_{20}$  is discussed in more detail below.

The Step 2 correction due to the  $K_1$  constituent is given below as an example.

$$\begin{aligned} (\Delta \bar{C}_{21} \times 10^{12})_{K_1} &= 507.4 \sin(\tau+s), \\ &= 507.4 \sin(\theta_g + \pi), \\ &= -507.4 \sin \theta_g. \end{aligned}$$

$$(\Delta \bar{S}_{21} \times 10^{12})_{K_1} = -507.4 \cos \theta_g.$$

The total variation in geopotential coefficients due to the solid tide is obtained by adding the results of Step 2 (Eq. 2) to those of Step 1 (Eq. 1).

Table 7.1

Step 2 Solid Tide Corrections When  $k_2 = .30$  in Step 1

Using a Cutoff Amplitude of  $9 \times 10^{12}$  for  $A_m \delta k_s H_s$

Long Period Tides (n=2, m=0)

None except zero frequency tide.

Diurnal Tides (n=2, m=1)

<u>Doodson Number</u>	$\bar{n}$ , argument multipliers						$A_m \delta k_s H_s \times 10^{12}$
	$\tau$	s	h	p	N'	$p_1$	
145.555 ( $O_1$ )	1	-1	0	0	0	0	-16.4
163.555 ( $P_1$ )	1	1	-2	0	0	0	-49.6
165.545	1	1	0	0	-1	0	-9.4
165.555 ( $K_1$ )	1	1	0	0	0	0	507.4
165.565	1	1	0	0	1	0	73.5
166.554 ( $\psi_1$ )	1	1	1	0	0	-1	-15.2

Semi-Diurnal Tides (n=2, m=2)

<u>Doodson Number</u>	$\bar{n}$ , argument multipliers						$A_m \delta k_s H_s \times 10^{12}$
	$\tau$	s	h	p	N'	$p_1$	
255.555 ( $M_2$ )	2	0	0	0	0	0	39.5
273.555 ( $S_2$ )	2	2	-2	0	0	0	18.4

Treatment of the Permanent Tide

The mean value of  $\Delta \bar{C}_{20}$  from Eq. 1a is not zero, and this permanent tide deserves special attention. The mean value of the correction could be included in the adopted value of  $C_{20}$  and hence not included in the  $\Delta C_{20}$ . The practical situation is not so clear because satellite derived values of  $C_{20}$  as in the GEM geopotentials have been obtained using a mixture of methods, some applying the corrections and others not applying it. There is no way to ensure consistency in this regard short of re-estimating  $C_{20}$  with a consistent technique. If this is done the inclusion of the zero frequency term in Eq. 1a should be avoided because  $k_2$  is not the appropriate Love number to use for such a term. The zero frequency change in  $C_{20}$  can be removed by computing  $\Delta C_{20}$  as

$$\Delta \bar{C}_{20}^* = \Delta \bar{C}_{20} (\text{Eq. 1a}) - \langle \Delta \bar{C}_{20} \rangle, \quad (4)$$

where

$$\begin{aligned} \langle \Delta \bar{C}_{20} \rangle &= A_0 H_0 k_2 \\ &= (4.4228 \times 10^{-8}) (-0.31455) k_2 \\ &= -1.39119 \times 10^{-8} k_2. \end{aligned}$$

Using  $k_2 = 0.30$  then

$$\langle \Delta \bar{C}_{20} \rangle = -4.1736 \times 10^{-9}$$

or

$$\langle \Delta J_2 \rangle = -\langle \Delta \bar{C}_{20} \rangle \sqrt{5} = 9.3324 \times 10^{-9}.$$

The decision to remove or not to remove the mean from the corrections depends on whether the adopted  $\bar{C}_{20}$  does or does not already contain it and on whether  $k_2$  is a potential 'solve for' parameter. If  $k_2$  is to be estimated then it must not multiply the zero frequency term in the correction. In the most recent data reductions leading to GEM-T3, the total tide correction was applied. If we assume the more recent data has most of the weight in the determination of  $\bar{C}_{20}$  then we conclude that the permanent deformation is not included in the GEM-T3 value of  $\bar{C}_{20}$ . Hence, if  $k_2$  is to be estimated, first  $\langle \Delta \bar{C}_{20} \rangle$  must be added to  $\bar{C}_{20}$  and then  $\Delta \bar{C}_{20}^*$  should be used in place of  $\Delta \bar{C}_{20}$  of Eq. 1a. The  $k_2$  used for restoring the permanent tide should match what was used in deriving the adopted value of  $\bar{C}_{20}$ .

The GEM-T3 value of  $\bar{C}_{20}$  is  $-484.16499 \times 10^{-6}$  and does not include the permanent deformation. The tidal corrections employed in the computations leading to GEM-T3 were equivalent to Eq. 1a with  $k_2 = 0.30$ . Let  $\bar{C}_{20}^*$  denote the coefficient which includes the zero frequency term; then the GEM-T3 values of  $\bar{C}_{20}$  with the permanent tide restored are:

$$\begin{aligned} \bar{C}_{20}^* (\text{GEM-T3}) &= -484.16499 \times 10^{-6} - (1.39119 \times 10^{-8}) \times 0.30, \\ \bar{C}_{20}^* (\text{GEM-T3}) &= -484.169164 \times 10^{-6}. \end{aligned} \quad (5)$$

These values for  $\bar{C}_{20}^*$  are recommended for use with the respective gravity field and should be added to the periodic tidal correction given as  $\Delta \bar{C}_{20}^*$  in Eq. 4 to get the total time dependent value of  $\bar{C}_{20}$ .

## Solid Tide Effect on Station Coordinates

The variations of station coordinates caused by solid Earth tides predicted using Wahr's theory are also most efficiently implemented using a two-step procedure. Only the second degree tides are necessary to retain 0.01 m precision. Also terms proportional to  $y$ ,  $h+$ ,  $h-$ ,  $z$ ,  $l+$ ,  $w+$ , and  $w-$  are ignored. The first step uses frequency independent Love and Shida numbers and a computation of the tidal potential in the time domain. A convenient formulation of the displacement is given in the documentation for the GEODYN program. The vector displacement of the station due to tidal deformation for Step 1 can be computed from

$$\Delta \vec{r} = \sum_{j=2}^3 \frac{[GM_j r^4]}{[GM_\oplus R_j^3]} \{ [3l_2 (\hat{R}_j \cdot \hat{r})] \hat{R}_j + [3(\frac{h_2}{2} - l_2) (\hat{R}_j \cdot \hat{r})^2 - \frac{h_2}{2}] \hat{r} \}, \quad (6)$$

$GM_j$  = gravitational parameter for the Moon ( $j=2$ ) or the Sun ( $j=3$ ),

$GM_\oplus$  = gravitational parameter for the Earth,

$\hat{R}_j, R_j$  = unit vector from the geocenter to Moon or Sun and the magnitude of that vector,

$\hat{r}, r$  = unit vector from the geocenter to the station and the magnitude of that vector,

$h_2$  = nominal second degree Love number,

$l_2$  = nominal Shida number.

If nominal values for  $h_2$  and  $l_2$  of 0.6090 and 0.0852 respectively are used with a cutoff of 0.005m of radial displacement, only one term needs to be corrected in Step 2. This is the  $K_1$  frequency where  $h$  from Wahr's theory is 0.5203. Only the radial displacement needs to be corrected and to sufficient accuracy this can be implemented as a periodic change in station height given by

$$\delta h_{STA} = \delta h_{K_1} H_{K_1} \left( -\sqrt{\frac{5}{24\pi}} \right) 3 \sin\phi \cos\phi \sin(\theta_{K_1} + \lambda), \quad (7)$$

where

$$\delta h_{K_1} = h_{K_1} \text{ (Wahr)} - h_2 \text{ (Nominal)} = -0.0887,$$

$H_{K_1}$  = amplitude of  $K_1$  term (165.555) in the harmonic expansion of the tide generating potential = 0.36878 m,

$\phi$  = geocentric latitude of station ,

$\lambda$  = east longitude of station ,

$\theta_{K_1} = K_1$  tide argument =  $\tau + s = \theta_z + \pi$ ,

or simplifying

$$\delta h_{STA} = -0.0253 \sin \phi \cos \phi \sin (\theta_z + \lambda).$$

The effect is maximum at  $\phi = 45^\circ$  where the amplitude is 0.013 m.

Equation (6) contains a site displacement that is independent of time. If nominal Love and Shida numbers of 0.6090 and 0.0852 respectively are used with Eq. 6, the permanent deformation introduced is in the radial direction.

$$\Delta \vec{r} \cdot \hat{r} = \sqrt{\frac{5}{4\pi}} (0.6090) (-0.31455) \left(\frac{3}{2} \sin^2 \phi - \frac{1}{2}\right) \quad (8a)$$

$$= -0.12083 \left(\frac{3}{2} \sin^2 \phi - \frac{1}{2}\right) \text{meters,}$$

and in the north direction

$$\Delta \vec{r} \cdot \hat{e}_p = \sqrt{\frac{5}{4\pi}} (0.0852) (-0.31455) 3 \cos \phi \sin \phi \quad (8b)$$

$$= -0.05071 \cos \phi \sin \phi \text{ meters.}$$

Since values for the Love numbers  $h_2$  and  $l_2$  substituted in (6) are representative only for short-period deformation, this permanent part has no physical significance and must be subtracted from (6). As a result, the tide model relates to the true long-time average positions of sites.

## Rotational Deformation Due to Polar Motion

The variation of station coordinates caused by the polar tide is recommended to be taken into account. Let us choose  $\hat{x}$ ,  $\hat{y}$ , and  $\hat{z}$  as a terrestrial system of reference. The  $\hat{z}$  axis is oriented along the Earth's mean rotation axis, the  $\hat{x}$  axis is in the direction of the adopted origin of longitude and  $\hat{y}$  axis is oriented along the  $90^\circ$  E meridian.

The centrifugal potential caused by the Earth's rotation is

$$V = \frac{1}{2} [r^2 |\bar{\Omega}|^2 - (\bar{r} \cdot \bar{\Omega})^2], \quad (9)$$

where  $\bar{\Omega} = \Omega(m_1 \hat{x} + m_2 \hat{y} + (1 + m_3) \hat{z})$ .  $\Omega$  is the mean angular velocity of rotation of the Earth,  $m_i$  are small dimensionless parameters,  $m_1$ ,  $m_2$  describing polar motion and  $m_3$  describing variation in the rotation rate,  $r$  is the radial distance to the station.

Neglecting the variations in  $m_3$  which induce displacements that are below the mm level, the  $m_1$  and  $m_2$  terms give a first order perturbation in the potential  $V$  (Wahr, 1985)

$$\Delta V(r, \theta, \lambda) = - \frac{\Omega^2 r^2}{2} \sin 2\theta (m_1 \cos \lambda + m_2 \sin \lambda), \quad (10)$$

where  $\theta$  is the co-latitude, and  $\lambda$  is the eastward longitude.

Let us define the radial displacement  $S_r$ , the horizontal displacements  $S_\theta$  and  $S_\lambda$ , positive upwards, south and east respectively, in a horizon system at the station due to  $\Delta V$  using the formulation of tidal Love numbers (W. Munk and G. MacDonald, 1960).

$$S_r = h \frac{\Delta V}{g},$$

$$S_\theta = \frac{\ell}{g} \partial_\theta \Delta V, \quad (11)$$

$$S_\lambda = \frac{\ell}{g} \frac{1}{\sin \theta} \partial_\lambda \Delta V,$$

where  $g$  is the gravitational acceleration at the Earth's surface,  $h$ ,  $\ell$  are the second-order body tide displacement Love Numbers.

In general, these computed displacements have a non-zero average over any given time span because  $m_1$  and  $m_2$ , used to find  $\Delta V$ , have a non-zero average. Consequently, the use of these results will lead to a change in the estimated mean station coordinates. When mean coordinates produced by different users are compared at the centimeter level, it is important to ensure that this effect has been handled consistently. It is recommended that  $m_1$  and  $m_2$  used in eq. 10 be replaced by parameters defined to be zero for the Terrestrial Reference Frame discussed in Chapter 3.

Thus, define

$$\begin{aligned} x_p &= m_1 - \bar{x}, \\ y_p &= -m_2 - \bar{y}, \end{aligned} \quad (12)$$



where  $\bar{x}$  and  $\bar{y}$  are the values of  $m_1$  and  $-m_2$  for the Chapter 3 Terrestrial Reference Frame. Then, using  $h = 0.6$ ,  $\ell = 0.085$ , and  $r = a = 6.4 \times 10^6 \text{m}$ ,

$$\begin{aligned} S_r &= -32 \sin 2\theta (x_p \cos \lambda - y_p \sin \lambda) \text{ mm}, \\ S_\theta &= -9 \cos 2\theta (x_p \cos \lambda - y_p \sin \lambda) \text{ mm}, \\ S_\lambda &= 9 \cos \theta (x_p \sin \lambda + y_p \cos \lambda) \text{ mm}. \end{aligned} \tag{13}$$

for  $x_p$  and  $y_p$  in seconds of arc.

Taking into account that  $x_p$  and  $y_p$  vary, at most, 0.8 arcsec, the maximum radial displacement is approximately 25 mm, and the maximum horizontal displacement is about 7 mm.

If  $X$ ,  $Y$ , and  $Z$  are Cartesian coordinates of a station in a right-hand equatorial coordinate system, we have the displacements of coordinates

$$[dX, dY, dZ]^T = R^T [S_\theta, S_\lambda, S_r]^T, \tag{14}$$

where

$$R = \begin{bmatrix} \cos \theta \cos \lambda & \cos \theta \sin \lambda & -\sin \theta \\ -\sin \lambda & \cos \lambda & 0 \\ \sin \theta \cos \lambda & \sin \theta \sin \lambda & \cos \theta \end{bmatrix}$$

The formula (13) can be used for determination of the corrections to station coordinates due to polar tide.

The deformation caused by the polar tide also leads to time-dependent perturbations in the  $C_{21}$  and  $S_{21}$  geopotential coefficients. The change in the external potential caused by this deformation is  $k_2 \Delta V$ , where  $\Delta V$  is given by eq. 10. Using  $k_2 = 0.30$  gives

$$\begin{aligned} \bar{C}_{21} &= -1.3 \times 10^{-9} (x_p), \\ \bar{S}_{21} &= -1.3 \times 10^{-9} (-y_p), \end{aligned}$$

where  $x_p$  and  $y_p$  are in seconds of arc and  $\bar{\phantom{x}}$  are used instead of  $m_1$  and  $-m_2$  so that no mean is introduced into  $\bar{C}_{21}$  and  $\bar{S}_{21}$  when making this correction.

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