CHAPTER 12 GENERAL RELATIVISTIC MODELS FOR PROPAGATION

VLBI Time Delay

There have been many papers dealing with relativistic effects which must be accounted for in VLBI processing; see (Robertson, 1975), (Finkelstein *et al.*, 1983), (Hellings, 1986), (Pavlov, 1985), (Cannon *et al.*, 1986), (Soffel *et al.*, 1986), (Zeller *et al.*, 1986), (Sovers and Fanselow, 1987), (Zhu and Groten, 1988), (Shahid-Saless *et al.*, 1991), (Soffel *et al.*, 1991). As pointed out by Boucher (1986), the relativistic correction models proposed in various articles are not quite compatible. To resolve differences between the procedures and to arrive at a standard model, a workshop was held at the U. S. Naval Observatory on 12 October 1990. The proceedings of this workshop have been published (Eubanks, 1991) and the model given here is based on the consensus model resulting from that workshop. Much of this chapter dealing with VLBI time delay is taken directly from that work and the reader is urged to consult that publication for further details. One change from that model has been made in order to adopt the IAU/IUGG conventions for the scale of the terrestrial reference system, in accord with 1991 resolutions (see Appendix in McCarthy, 1992). Geodetic lengths should be expressed in "SI units," *i.e.*, be consistent with the second as realized by a clock running at the TAI rate at sea level. The only change needed to the 1992 formulation to satisfy the IAU/IUGG Resolutions is to include the Earth's potential, $U_\oplus$, in the total potential U in the delay equation (Equation 9). The only observable effect of this change will be an increase of the VLBI terrestrial scale by 1.39385806 parts in 10^9.

As pointed out by Eubanks, the use of clocks running at the geoid and delays calculated "at the geocenter" ignoring the scale change induced by the Earth's gravitational potential means that terrestrial distances calculated from the consensus model will not be the same as those calculated using the expressions in this chapter which are equivalent to using meter sticks on the surface of the Earth. The accuracy limit chosen for the consensus VLBI relativistic delay model is 10^{-12} seconds (one picosecond) of differential VLBI delay for baselines less than two Earth radii in length. In the model all terms of order 10^{-13} seconds or larger were included to ensure that the final result was accurate at the picosecond level. Source coordinates derived from the consensus model will be solar system barycentric and should have no apparent motions due to solar system relativistic effects at the picosecond level.

The consensus model was derived from a combination of five different relativistic models for the geodetic delay. These are the Masterfit/Modest model, due to Fanselow and Thomas (see Treuhaft and Thomas, in (Eubanks, 1991), and (Sovers and Fanselow, 1987)), the I. I. Shapiro model (see Ryan, in (Eubanks, 1991)), the Hellings-Shahid-Saless model (Shahid-Saless *et al.*, 1991) and in (Eubanks, 1991), the Soffel, Muller, Wu and Xu model (Soffel *et al.*, 1991) and in (Eubanks, 1991), and the Zhu-Groten model (Zhu and Groten, 1988) and in (Eubanks, 1991). Baseline results are expressed in "local" or "SI" coordinates appropriate for clocks running at the TAI rate on the surface of the Earth, to be consistent with the more general IERS conventions on the terrestrial reference system scale. This means that the gravitational potential of the Earth is now included in U; the scale effects of the geocentric station velocities can still be ignored as they will at most cause scale changes of order 1 part in 10^{12} (see Zhu and Groten, Soffel *et al.*, and Fukushima, all in Eubanks (1991) and Shahid-Saless *et al.* (1991) for further details on the implications of these choices). As the time argument is now based on TAI, which is a quasi-local time on the geoid, distance estimates from these conventions will now be consistent in principle with "physical" distances.

The model is designed for use in the reduction of VLBI observations of extra-galactic objects acquired from the surface of the Earth. The delay error caused by ignoring the annual parallax is > 1 psec for objects closer than several hundred thousand light years, which includes all of the Milky Way galaxy. The model is not intended for use with observations of sources in the solar system, nor is it intended for use with observations made from space-based VLBI, from either low or high Earth orbit, or from the surface of the Moon (although it would be suitable with obvious changes for observations made entirely from the Moon).

It is assumed that the inertial reference frame is defined kinematically and that very distant objects, showing no apparent motion, are used to estimate precession and the nutation series. This frame is not truly inertial in a dynamical sense, as included in the precession constant and nutation series are the effects of the geodesic precession ($\sim$ 19 milli arc seconds / year). Soffel *et al.* (in Eubanks (1991)) and Shahid-Saless *et al.* (1991) give details of a dynamically inertial VLBI delay equation. At the picosecond level, there is no practical difference for VLBI geodesy and astrometry except for the adjustment in the precession constant.

Although the delay to be calculated is the time of arrival at station 2 minus the time of arrival at station 1, it is the time of arrival at station 1 that serves as the time reference for the measurement. Unless explicitly stated otherwise, all vector and scalar quantities are assumed to be calculated at t_1, the time of arrival at station 1 including the effects of the troposphere.

The notation follows that of Hellings (1986) and Hellings and Shahid-Saless in Eubanks (1991) as closely as possible. It is assumed that the standard IAU models for precession, nutation, Earth rotation and polar motion have been followed and that all geocentric vector quantities have thus been rotated into a nearly non-rotating celestial frame. The errors in the standard IAU models are negligible for the purposes of the relativistic transformations. The notation itself is given in Table 12.1. The consensus model separates the total delay into a classical delay and a general relativistic delay, which are then modified by relativistic transformations between geocentric and solar system barycentric frames.

Table 12.1. Notation used in the model

t_i	the time of arrival of a radiointerferometric signal at the i^{th} VLBI receiver in terrestrial time (TAI)
T_i	the time of arrival of a radiointerferometric signal at the i^{th} VLBI receiver in barycentric time (TCB or TDB)
t_{g_i}	the "geometric" time of arrival of a radiointerferometric signal at the i^{th} VLBI receiver including the gravitational "bending" delay and the change in the geometric delay caused by the existence of the atmospheric propagation delay but neglecting the atmospheric propagation delay itself
t_{v_i}	the "vacuum" time of arrival of a radiointerferometric signal at the i^{th} VLBI receiver including the gravitational delay but neglecting the atmospheric propagation delay and the change in the geometric delay caused by the existence of the atmospheric propagation delay
t_{i_J}	the approximation to the time that the ray path to station i passed closest to gravitating body J
δt_{atm_i}	the atmospheric propagation delay for the i^{th} receiver $= t_i - t_{g_i}$
Δt_{grav}	the differential gravitational time delay, commonly known as the gravitational "bending delay"
$\vec{x}_i(t_i)$	the geocentric radius vector of the i^{th} receiver at the geocentric time t_i
$\vec{b}$	$\vec{x}_2(t_1) - \vec{x}_1(t_1)$ and is thus the geocentric baseline vector at the time of arrival t_1
$\vec{b}_0$	the *a priori* geocentric baseline vector at the time of arrival t_1
$\delta\vec{b}$	$\vec{b}(t_1) - \vec{b}_0(t_1)$

$\vec{w}_i$ the geocentric velocity of the i^{th} receiver
$\hat{K}$ the unit vector from the barycenter to the source in the absence of gravitational or aberrational bending
$\hat{k}_i$ the unit vector from the i^{th} station to the source after aberration
$\vec{X}_i$ the barycentric radius vector of the i^{th} receiver
$\vec{X}_\oplus$ the barycentric radius vector of the geocenter
$\vec{X}_J$ the barycentric radius vector of the J^{th} gravitating body
$\vec{R}_{i_J}$ the vector from the J^{th} gravitating body to the i^{th} receiver
$\vec{R}_{\oplus_J}$ the vector from the J^{th} gravitating body to the geocenter
$\vec{R}_{\oplus_\odot}$ the vector from the Sun to the geocenter
$\hat{N}_{i_J}$ the unit vector from the J^{th} gravitating body to the i^{th} receiver
$\vec{V}_\oplus$ the barycentric velocity of the geocenter
U the gravitational potential at the geocenter plus the terrestrial potential at the surface of the Earth. At the picosecond level, only the solar and terrestrial potentials need be included in U so that $U = GM_\odot/|\vec{R}_{\oplus_\odot}|c^2 + GM_\oplus/a_\oplus c^2$
M_i the mass of the i^{th} gravitating body
$M_\oplus$ the mass of the Earth
γ a PPN Parameter = 1 in general relativity
c the speed of light in meters / second
G the Gravitational Constant in Newtons meters^2 kilograms^{-2}
$a_\oplus$ the equatorial radius of the Earth

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Vector magnitudes are expressed by the absolute value sign $[|x| = (\Sigma x_i^2)^{\frac{1}{2}}]$. Vectors and scalars expressed in geocentric coordinates are denoted by lower case (*e.g.* $\vec{x}$ and t), while quantities in barycentric coordinates are in upper case (*e.g.* $\vec{X}$ and T). MKS units are used throughout. For quantities such as $V_\oplus$, $\vec{w}_i$, and U it is assumed that a table (or numerical formula) is available as a function of TAI and that they are evaluated at the atomic time of reception at station 1, t_1, unless explicitly stated otherwise. A lower case subscript (*e.g.* $\vec{x}_i$) denotes a particular VLBI receiver, while an upper case subscript (*e.g.* $\vec{x}_J$) denotes a particular gravitating body.

Gravitational Delay

The general relativistic delay, Δt_{grav}, is given for the J^{th} gravitating body by

$$\Delta t_{grav_J} = (1+\gamma)\frac{GM_J}{c^3}\ln\frac{|\vec{R}_{1_J}| + \vec{K}\cdot\vec{R}_{1_J}}{|\vec{R}_{2_J}| + \vec{K}\cdot\vec{R}_{2_J}}. \tag{1}$$

At the picosecond level it is possible to simplify the delay due to the Earth, $\Delta t_{grav_\oplus}$, which becomes

$$\Delta t_{grav_\oplus} = (1+\gamma)\frac{GM_\oplus}{c^3}\ln\frac{|\vec{x}_1| + \vec{K}\cdot\vec{x}_1}{|\vec{x}_2| + \vec{K}\cdot\vec{x}_2}. \tag{2}$$

The Sun, the Earth and Jupiter must be included, as well as the other planets in the solar system along with the Earth's Moon, for which the maximum delay change is several picoseconds. The major satellites of Jupiter, Saturn and Neptune should also be included if the ray path passes close to them. This is very unlikely in normal geodetic observing but may occur during planetary occultations.

The effect on the bending delay of the motion of the gravitating body during the time of propagation along the ray path is small for the Sun but can be several hundred picoseconds for Jupiter (see Sovers and Fanselow (1987) page 9). Since this simple correction, suggested by Sovers and Fanselow (1987) and Hellings (1986) among others, is sufficient at the picosecond level, it was adapted for the consensus model. It is also necessary to account for the motion of station 2 during the propagation time between station 1 and station 2. In this model $\vec{R}_{i_J}$, the vector from the J^{th} gravitating body to the i^{th} receiver, is iterated once, giving

$$t_{1_J} = \min[t_1, t_1 - \hat{K} \cdot (\vec{X}_J(t_1) - \vec{X}_1(t_1))], \tag{3}$$

so that

$$\vec{R}_{1_J}(t_1) = \vec{X}_1(t_1) - \vec{X}_J(t_{1_J}), \tag{4}$$

and

$$\vec{R}_{2_J} = \vec{X}_2(t_1) - \frac{\vec{V}_\oplus}{c}(\hat{K} \cdot \vec{b}_0) - \vec{X}_J(t_{1_J}). \tag{5}$$

Only this one iteration is needed to obtain picosecond level accuracy for solar system objects. If more accuracy is required, it is probably better to use the rigorous approach of Shahid-Saless *et al.* (1991). $\vec{X}_1(t_1)$ is not tabulated, but can be inferred from $\vec{X}_\oplus(t_1)$ using

$$\vec{X}_i(t_1) = \vec{X}_\oplus(t_1) + \vec{x}_i(t_1), \tag{6}$$

which is of sufficient accuracy for use in equations 3, 4, and 5, when substituted into equation 1 but not for use in computing the geometric delay. The total gravitational delay is the sum over all gravitating bodies including the Earth,

$$\Delta t_{grav} = \sum_J \Delta t_{grav_J}. \tag{7}$$

Geometric Delay

In the barycentric frame the vacuum delay equation is, to a sufficient level of approximation:

$$T_2 - T_1 = -\frac{1}{c}\hat{K} \cdot (\vec{X}_2(T_2) - \vec{X}_1(T_1)) + \Delta t_{grav}. \tag{8}$$

This equation is converted into a geocentric delay equation using known quantities by performing the relativistic transformations relating the barycentric vectors $\vec{X}_i$ to the corresponding geocentric vectors $\vec{x}_i$, thus converting equation 8 into an equation in terms of $\vec{x}_i$. The related transformation between barycentric and geocentric time can be used to derive another equation relating $T_2 - T_1$ and $t_2 - t_1$, and these two equations can then be solved for the geocentric delay in terms of the geocentric baseline vector $\vec{b}$. The papers by Soffel *et al.* in Eubanks (1991), Hellings and Shahid-Saless in Eubanks (1991), Zhu and Groten (1988) and Shahid-Saless *et al.* (1991) give details of the derivation of the vacuum delay equation. To conserve accuracy and simplify the equations the delay was expressed as much as is possible in terms of a rational polynomial. In the rational polynomial form the total geocentric vacuum delay is given by

$$t_{v_2} - t_{v_1} = \frac{\Delta t_{grav} - \frac{\hat{K}\cdot\vec{b}_0}{c}[1 - (1+\gamma)U - \frac{|\vec{V}_\oplus|^2}{2c^2} - \frac{\vec{V}_\oplus\cdot\vec{w}_2}{c^2}] - \frac{\vec{V}_\oplus\cdot\vec{b}_0}{c^2}(1 + \hat{K}\cdot\vec{V}_\oplus/2c)}{1 + \frac{\hat{K}\cdot(\vec{V}_\oplus + \vec{w}_2)}{c}}. \tag{9}$$

Given this expression for the vacuum delay, the total delay is found to be

$$t_2 - t_1 = t_{v_2} - t_{v_1} + (\delta t_{atm_2} - \delta t_{atm_1}) + \delta t_{atm_1} \frac{\hat{K} \cdot (\vec{w}_2 - \vec{w}_1)}{c}. \tag{10}$$

For convenience the total delay can be divided into separate geometric and propagation delays. The geometric delay is given by

$$t_{g_2} - t_{g_1} = t_{v_2} - t_{v_1} + \delta t_{atm_1} \frac{\hat{K} \cdot (\vec{w}_2 - \vec{w}_1)}{c}, \tag{11}$$

and the total delay can be found at some later time by adding the propagation delay:

$$t_2 - t_1 = t_{g_2} - t_{g_1} + (\delta t_{atm_2} - \delta t_{atm_1}). \tag{12}$$

The tropospheric propagation delay in equations 11 and 12 need not be from the same model. The estimate in equation 12 should be as accurate as possible, while the δt_{atm} model in equation 11 need only be accurate to about an air mass ($\sim$ 10 nanoseconds). If equation 10 is used instead, the model should be as accurate as is possible.

If the difference, $\delta\vec{b}$, between the *a priori* baseline vector $\vec{b_0}$ used in equation 9 and the true baseline vector is less than roughly three meters, then it suffices to add $-(\hat{K} \cdot \delta\vec{b})/c$ to $t_2 - t_1$. If this is not the case, however, the delay must be modified by adding

$$\Delta(t_{g_2} - t_{g_1}) = -\frac{\frac{\hat{K} \cdot \delta\vec{b}_0}{c}}{1 + \frac{\hat{K} \cdot (\vec{V}_\oplus + \vec{w}_2)}{c}} - \frac{\vec{V}_\oplus \cdot \delta\vec{B}}{c^2} \tag{13}$$

to the total time delay $t_2 - t_1$ from equation 10 or 12.

Observations Close to the Sun

For observations made very close to the Sun, higher order relativistic time delay effects become increasingly important. The largest correction is due to the change in delay caused by the bending of the ray path by the gravitating body described in Richter and Matzner (1983) and Hellings (1986). The change to Δt_{grav} is

$$\delta t_{grav_i} = \frac{(1+\gamma)^2 G^2 M_i^2}{c^5} \frac{\vec{b} \cdot (\hat{N}_{1_i} + \hat{K})}{(|\vec{R}|_{1_i} + \vec{R}_{1_i} \cdot \hat{K})^2}, \tag{14}$$

which should be added to the Δt_{grav} in equation 1.

Summary

Assuming that time t_1 is the Atomic (TAI) time of reception of the VLBI signal at receiver 1, the following steps are recommended to correct the VLBI time delay for relativistic effects.

1. Use equation 6 to estimate the barycentric station vector for receiver 1.

2. Use equations 3, 4, and 5 to estimate the vectors from the Sun, the Moon, and each planet except the Earth to receiver 1.

3. Use equation 1 to estimate the differential gravitational delay for each of those bodies.

4. Use equation 2 to find the differential gravitational delay due to the Earth.

5. Sum to find the total differential gravitational delay.

6. Add Δt_{grav} to the rest of the *a priori* vacuum delay from equation 9.

7. Calculate the aberrated source vector for use in the calculation of the tropospheric propagation delay:

$$\vec{k}_i = \hat{K} + \frac{\vec{V}_\oplus + \vec{w}_i}{c} - \hat{K}\frac{\hat{K} \cdot (\vec{V}_\oplus + \vec{w}_i)}{c}. \tag{15}$$

8. Add the geometric part of the tropospheric propagation delay to the vacuum delay, equation 11.

9. The total delay can be found by adding the best estimate of the tropospheric propagation delay

$$t_2 - t_1 = t_{g_2} - t_{g_1} + [\delta t_{atm_2}(t_1 - \frac{\hat{K} \cdot \vec{b}_0}{c}, \vec{k}_2) - \delta t_{atm_1}(\vec{k}_1)]. \tag{16}$$

10. If necessary, apply equation 13 to correct for "post-model" changes in the baseline by adding equation 13 to the total time delay from equation step 9.

Propagation Correction for Laser Ranging

The space-time curvature near a massive body requires a correction to the Euclidean computation of range, ρ. This correction in seconds, Δt, is given by (Holdridge, 1967)

$$\Delta t = \frac{(1+\gamma)GM}{c^3} \ln\left(\frac{R_1 + R_2 + \rho}{R_1 + R_2 - \rho}\right), \tag{17}$$

where

c = speed of light,
γ = PPN parameter equal to 1 in General Relativity,
R_1 = distance from the body's center to the beginning of the light path,
R_2 = distance from the body's center to the end of the light path,
GM = gravitational parameter of the deflecting body.

For near-Earth satellites, working in the geocentric frame of reference, the only body to be considered is the Earth (Ries *et al.*, 1989). For lunar laser ranging, which is formulated in the solar system barycentric reference frame, the Sun and the Earth must be considered (Newhall *et al.*, 1987).

In the computation of the instantaneous space-fixed positions of a station and a lunar reflector in the analysis of LLR data, the body-centered coordinates of the two sites are affected by a scale reduction and a Lorentz contraction effect (Martin *et al.*, 1985). The scale effect is about 15 cm in the height of a tracking station, while the maximum value of the Lorentz effect is about 3 cm.

The equation for the transformation of $\vec{r}$, the geocentric position vector of a station expressed in the geocentric frame, is

$$\vec{r}_b = \vec{r}\left(1 - \frac{\gamma\Phi}{c^2}\right) + \frac{1}{2}\left(\frac{\vec{V}\cdot\vec{r}}{c^2}\right)\vec{V}, \tag{18}$$

where

$\vec{r}_b$ = station position expressed in the barycentric frame,
Φ = gravitational potential at the geocenter (excluding the Earth's mass),
$\vec{V}$ = barycentric velocity of the Earth,

A similar equation applies to the selenocentric reflector coordinates; the maximum value of the Lorentz effect is about 1 cm (Newhall *et al.*, 1987).

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