I. Current methodology for TRF combination

The current methodology is based on combining simultaneously station positions and velocities using full variance-covariance information provided, in SINEX format, by the IERS analysis centers. Moreover, a rigorous weighting scheme is used, based on the analysis and estimation of the variance components using Helmert method. We recall briefly hereafter the ITRF physical combination as well as statistical modelling and other TRF investigation aspects such as orthogonal projection and minimal position variance.

I.1. Physical model for simultaneous combination of station positions and velocities

Assuming that for each individual solution \( s \), and each point \( i \), we have position \( X_i^s \) at epoch \( t_i^s \) and velocity \( \dot{X}_i^s \), expressed in a given TRF \( k \).

The combination consists on estimating:

- Positions \( X_i^{itr} \) at a given epoch \( t_0 \) and velocities \( \dot{X}_i^{itr} \) in ITRS
- Transformation parameters \( T_k \) at an epoch \( t_k \) and their rates \( \dot{T}_k \), from the ITRF to each individual frame \( k \)

The general physical model used is given by the following equation (I.1):

\[
\begin{align*}
X_i^s &= X_i^{itr} + (t_i^s - t_0) \dot{X}_i^{itr} + T_k + D_k X_i^{itr} + R_k X_i^{itr} \\
\dot{X}_i^s &= \dot{X}_i^{itr} + \dot{T}_k + \dot{D}_k X_i^{itr} + \dot{R}_k X_i^{itr} \\
\end{align*}
\]

(I.1)

where for each individual frame \( k \), \( D_k \) is the scale factor, the translation vector \( T_k \) and rotation matrix \( R_k \) are respectively defined (following IERS conventions) by:

\[
T_k = \begin{pmatrix} T_{1k} \\ T_{2k} \\ T_{3k} \end{pmatrix} \quad \text{and} \quad R_k = \begin{pmatrix} 0 & -R_3 & R_2 \\ R_3 & 0 & -R_1 \\ -R_2 & R_1 & 0 \end{pmatrix}
\]

The dotted parameters designate their derivatives with respect to time. \( T_1, T_2, T_3 \) are the 3 origin components, \( R_1, R_2, R_3 \) are the three small rotations according to the 3 axes, respectively \( X, Y, Z \).
I.2. Statistical and variance estimation modeling

Combining heterogeneous data has been, since the beginning of ITRF combinations, one of the key element of ITRF analysis. The problem is, by essence, statistic. Up to now, the data were homogenized by the use of an empirical weighting scheme. For the ITRF96, a theoretical estimate of weights has been built and used in a rigorous stochastic framework.

The individual solutions, as well as the eccentricities are considered to be statistically independent populations. In addition, their variance matrix is supposed to be known except from a multiplicative factor (the variance factor). Therefore, for the $\ell$th set, we can postulate:

\[
\begin{align*}
E(Y_\ell) &= Y_\ell \\
\text{cov}(Y_\ell, Y_\ell) &= \delta_\ell^2 \sigma_\ell^2 \Sigma_\ell
\end{align*}
\] (1.2)

where $Y_\ell$ is the vector of observations (coordinates and velocities) in the $\ell$th set and $\delta_\ell^2$ is the Kronecker symbol ($1 \leq \ell \leq p$). The variance factor $\sigma_\ell^2$ is unknown and estimated iteratively through the Helmert-variance components estimation technique (Graafarend, 1984; Rao, 1973; Rao and Kleffe, 1988). The estimate $\hat{s}$ of the vector $s = (\sigma_1^2, \ldots, \sigma_p^2)^T$ is given by:

\[
\begin{align*}
\hat{s} &= \left( \begin{array}{c}
\hat{\sigma}_1^2 \\
\vdots \\
\hat{\sigma}_p^2 
\end{array} \right) = H^{-1}q \\
H_{\ell,k} &= \delta_\ell^2 [n_\ell - 2tr(vN_\ell)] + tr(vN_\ell vN_k) \\
q_\ell &= v_\ell^T \Sigma_\ell^{-1} v_\ell
\end{align*}
\] (1.3)

where $tr$ is the function trace and $\nu$ is the inverse of normal matrix of the whole adjustment. $N_\ell$ is the normal matrix of the $\ell$th solution and for which $n_\ell$ is the total number of observations; $v_\ell$ is the position and velocity residual vector of the combination. $\hat{s}$ is an unbiased estimate of $s$ and the basic idea of constructing such estimator is to minimize, in the sense of the Gauss-Markov theorem, the variance of the resulting least square estimate ($\nu$).

I.3. Orthogonal projection

In some solutions and in particular those derived with "free network" approach, the underlying reference system is loosely fixed and consequently this modifies the quality assessment of the individual solutions included in the combination. For a given solution of positions and velocities, $X_\ell$, with its associated covariance matrix $\Sigma_\ell$, this latter contains both the natural observation noise and the Reference System Effect (RSE). The orthogonal projection consists in replacing, in $\Sigma_\ell$, the original RSE by a well known and small RSE, for all individual Solutions to be combined together. This could be achieved by constructing an orthogonal projector $B$ as:

\[
B = (A^T A)^{-1} A^T
\] (1.4)

where:

- $A$ is the matrix of partial derivatives:

\[
A = \begin{pmatrix}
1 & 0 & 0 & x_0^0 & 0 & z_0^0 & -y_0^0 \\
0 & 1 & 0 & y_0^0 & -x_0^0 & 0 & z_0^0 \\
0 & 0 & 1 & x_0^0 & y_0^0 & -z_0^0 & 0 \\
0 & 0 & 0 & 1 & y_0^0 & -z_0^0 & -x_0^0 \\
0 & 0 & 0 & 0 & 1 & x_0^0 & y_0^0 & -z_0^0 \\
0 & 0 & 0 & 0 & 0 & 1 & y_0^0 & -z_0^0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & y_0^0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
\]

$I(Y)$ is the expectancy of the random variable $Y$ and $Y$ is the exact value of $Y$; $\text{cov}(X, Y)$ is the covariance matrix of $X$ and $Y$. 

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• \((x_i^0, y_i^0, z_i^0)\) are the approximate coordinates of each point \(i\)

Using the orthogonal projector (1.4), \(\Sigma_s\) is modified by:

\[
\overline{\Sigma}_s \leftarrow \Sigma_s - \Sigma_s B^T (B\Sigma_s B^T + \Sigma_\theta)^{-1} B\Sigma_s
\]

(1.5)

where \(\theta\) is the vector of the transformation parameters; \(\theta = (T_x, T_y, T_z, D, R_x, R_y, R_z, T_\varphi \ldots)^T\) and \(\Sigma_\theta\) is a diagonal matrix containing small variances for each transformation parameter.

### I.4. Minimal Position Variance Investigation

It is usual to consider that the appropriate epoch to which station positions should be referred, is the "central epoch" of the used observations. This is theoretically true if the observations are regular and have the same quality in time. On the other hand, if positions and velocities are solved for, together with full variance/covariance between them, then the corresponding matrix should reflect the overall quality (or internal consistency) of the solution. Thus, propagating the solution (positions and variances) at any reasonable epoch (within the time interval of the used observations) should not, in principle, disturb the internal consistency of the solution.

In their submissions to ITRF, almost all the individual analysis centers adopt a unique epoch for all station positions. Meanwhile, it could sometimes occur that, due to some data sampling reasons, the selected unique epoch degrade the inner quality of the solution. Therefore, to overcome this possibility, it is also necessary to investigate whether the entire solution becomes more precise if the positions are propagated at their Epochs of Minimal Position Variance (EMPV). This latter could be computed for each point \(i\) by:

\[
EMPV_i = \min_i [\text{tr}(\text{var}(X^i(t)))]
\]

(1.6)

Using the general combination model, where \(X^i\) is the same as defined in (1.1), and in terms of the estimated parameters, combining solutions propagated at their EMPV's or at a unique epoch, is fully equivalent.