6 Geopotential

Gravitational models commonly used in current (2003) precision orbital analysis by contributors to the International Laser Ranging Service (ILRS) include EGM96 (Lemoine et al., 1998), JGM-3 (Tapley et al., 1996), and GRIM5-C1 (Gruber et al., 2000). For products of interest to IERS, similar accuracy is achievable with any of these models. IERS, recognizing the continuous development of new gravitational models, and anticipating the results of upcoming geopotential mapping missions, recommends at this time the EGM96 model as the conventional model. The GM$_{\oplus}$ and $\alpha_e$ values reported with EGM96 (398600.4415 km$^3$/s$^2$ and 6378136.3 m) should be used as scale parameters with the geopotential coefficients. The recommended GM$_{\oplus} = 398600.4418$ should be used with the two-body term when working with Geocentric Coordinate Time (TCG) (398600.4415 or 398600.4356 should be used by those still working with Terrestrial Time (TT) or Barycentric Dynamical Time (TDB) units, respectively). EGM96 is available at \footnote{http://www.nima.mil/GandG/wgsegm/egm96.html}. Values for the $C_{21}$ and $S_{21}$ coefficients are included in the EGM96 model. The $C_{21}$ and $S_{21}$ coefficients describe the position of the Earth’s figure axis. When averaged over many years, the figure axis should closely coincide with the observed position of the rotation pole averaged over the same time period. Any differences between the mean figure and mean rotation pole averaged would be due to long-period fluid motions in the atmosphere, oceans, or Earth’s fluid core (Wahr, 1987; 1990). At present, there is no independent evidence that such motions are important. The EGM96 values for $C_{21}$ and $S_{21}$ give a mean figure axis that corresponds to the mean pole position recommended in Chapter 4 Terrestrial Reference Frame.

This choice for $C_{21}$ and $S_{21}$ is realized as follows. First, to use the geopotential coefficients to solve for a satellite orbit, it is necessary to rotate from the Earth-fixed frame, where the coefficients are pertinent, to an inertial frame, where the satellite motion is computed. This transformation between frames should include polar motion. We assume the polar motion parameters used are relative to the IERS Reference Pole. If $\bar{x}$ and $\bar{y}$ are the angular displacements of the pole of the Terrestrial Reference Frame described in Chapter 4 relative to the IERS Reference Pole, then the values

\[
\begin{align*}
\bar{C}_{21} &= \sqrt{3} \bar{x} C_{20} - \bar{x} C_{22} + \bar{y} S_{22}, \\
\bar{S}_{21} &= -\sqrt{3} \bar{y} C_{20} - \bar{y} C_{22} - \bar{x} S_{22},
\end{align*}
\]

where $\bar{x} = 0.262 \times 10^{-6}$ radians (equivalent to 0.054 arcsec) and $\bar{y} = 1.730 \times 10^{-6}$ radians (equivalent to 0.357 arcsec) are those determined from observations available from the IERS at \footnote{http://maia.usno.navy.mil/conv2000/chapter7/annual.pole}, so that the mean figure axis coincides with the pole described in Chapter 4.

The EGM96 values at 1 January 2000 are $\bar{C}_{20} = -4.84165209 \times 10^{-4}$ (tide free), $\bar{C}_{20} = -4.84169382 \times 10^{-4}$ (zero tide), and $d\bar{C}_{20}/dt = +1.162755 \times 10^{-11}$/year.

This gives normalized coefficients of

\[
\begin{align*}
\bar{C}_{21}(\text{IERS}) &= -2.23 \times 10^{-10}, \text{ and} \\
\bar{S}_{21}(\text{IERS}) &= 14.48 \times 10^{-10}.
\end{align*}
\]

$\bar{C}_{21}$ and $\bar{S}_{21}$ are time variable. The values above are associated with the epoch of 1 January 2000. The complete definition of the instantaneous values of the two coefficients to use when computing orbits is given by:

\[
\begin{align*}
\bar{C}_{21} &= \bar{C}_{21}(t_0) + d\bar{C}_{21}/dt[t - t_0], \text{ and} \\
\bar{S}_{21} &= \bar{S}_{21}(t_0) + d\bar{S}_{21}/dt[t - t_0],
\end{align*}
\]
where \(d\bar{C}_{21}/dt\) and \(d\bar{S}_{21}/dt\) are the time derivatives determined at epoch \(t_0\) to be \(-0.337 \times 10^{-11}/y\) and \(+1.606 \times 10^{-11}/y\) respectively. It is also necessary to account for the solid Earth pole tide described later in this chapter.

### 6.1 Effect of Solid Earth Tides

The changes induced by the solid Earth tides in the free space potential are most conveniently modeled as variations in the standard geopotential coefficients \(C_{nm}\) and \(S_{nm}\) (Eanes et al., 1983). The contributions \(\Delta C_{nm}\) and \(\Delta S_{nm}\) from the tides are expressible in terms of the \(k\) Love number. The effects of ellipticity and of the Coriolis force due to Earth rotation on tidal deformations necessitates the use of three \(k\) parameters, \(k^{(0)}_{nm}\) and \(k^{(\pm)}_{nm}\) (except for \(n = 2\)) to characterize the changes produced in the free space potential by tides of spherical harmonic degree and order \((nm)\) (Wahr, 1981); only two parameters are needed for \(n = 2\) because \(k^{(-)}_{2m} = 0\) is zero due to mass conservation.

Anelasticity of the mantle causes \(k^{(0)}_{nm}\) and \(k^{(\pm)}_{nm}\) to acquire small imaginary parts (reflecting a phase lag in the deformational response of the Earth to tidal forces), and also gives rise to a variation with frequency which is particularly pronounced within the long period band. Though modeling of anelasticity at the periods relevant to tidal phenomena (8 hours to 18.6 years) is not yet definitive, it is clear that the magnitudes of the contributions from anelasticity cannot be ignored (see below). Recent evidence relating to the role of anelasticity in the accurate modeling of nutation data (Mathews et al., 2002) lends support to the model employed herein, at least up to diurnal tidal periods; and there is no compelling reason at present to adopt a different model for the long period tides.

Solid Earth tides within the diurnal tidal band (for which \((nm) = (21)\)) are not wholly due to the direct action of the tide generating potential (TGP) on the solid Earth; they include the deformations (and associated geopotential changes) arising from other effects of the TGP, namely, ocean tides and wobbles of the mantle and the core regions. Deformation due to wobbles arises from the incremental centrifugal potentials caused by the wobbles; and ocean tides load the crust and thus cause deformations. Anelasticity affects the Earth’s deformational response to all these types of forcing.

The wobbles, in turn, are affected by changes in the Earth’s moment of inertia due to deformations from all sources, and in particular, from the deformation due to loading by the \((nm) = (21)\) part of the ocean tide; wobbles are also affected by the anelasticity contributions to all deformations, and by the coupling of the fluid core to the mantle and the inner core through the action of magnetic fields at its boundaries (Mathews et al., 2002). Resonances in the wobbles—principally, the Nearly Diurnal Free Wobble resonance associated with the Free Core Nutation (FCN)—and the consequent resonances in the contribution to tidal deformation from the centrifugal perturbations associated with the wobbles, cause the body tide and load Love/Shida number parameters of the diurnal tides to become strongly frequency dependent. For the derivation of resonance formulae of the form (6) below to represent this frequency dependence, see Mathews et al., (1995). The resonance expansions assume that the Earth parameters entering the wobble equations are all frequency independent. However the ocean tide induced deformation makes a frequency dependent contribution to deformability parameters which are among the Earth parameters just referred to. It becomes necessary therefore to add small corrections to the Love number parameters computed using the resonance formulae. These corrections are included.
in the tables of Love number parameters given in this chapter and the next.

The deformation due to ocean loading is itself computed in the first place using frequency independent load Love numbers (see the penultimate section of this chapter and the first section of Chapter 7). Corrections to take account of the resonances in the load Love numbers are incorporated through equivalent corrections to the body tide Love numbers, following Wahr and Sasao (1981), as explained further below. These corrections are also included in the tables of Love numbers.

The degree 2 tides produce time dependent changes in $C_{2m}$ and $S_{2m}$, through $k_{2m}^{(0)}$, which can exceed $10^{-8}$ in magnitude. They also produce changes exceeding $3 \times 10^{-12}$ in $C_{4m}$ and $S_{4m}$ through $k_{2m}^{(+)}$. (The direct contributions of the degree 4 tidal potential to these coefficients are negligible.) The only other changes exceeding this cutoff are in $C_{3m}$ and $S_{3m}$, produced by the degree 3 part of the tide generating potential.

The computation of the tidal contributions to the geopotential coefficients is most efficiently done by a three-step procedure. In Step 1, the $(2m)$ part of the tidal potential is evaluated in the time domain for each $m$ using lunar and solar ephemerides, and the corresponding changes $\Delta C_{2m}$ and $\Delta S_{2m}$ are computed using frequency independent nominal values $k_{2m}^{(0)}$ for the respective $k_{2m}^{(0)}$. The contributions of the degree 3 tides to $C_{3m}$ and $S_{3m}$ through $k_{3m}^{(0)}$ and also those of the degree 2 tides to $C_{4m}$ and $S_{4m}$ through $k_{2m}^{(+)}$ may be computed by a similar procedure; they are at the level of $10^{-11}$.

Step 2 corrects for the deviations of the $k_{21}$ of several of the constituent tides of the diurnal band from the constant nominal value $k_{21}$ assumed for this band in the first step. Similar corrections need to be applied to a few of the constituents of the other two bands also.

Steps 1 and 2 can be used to compute the total tidal contribution, including the time independent (permanent) contribution to the geopotential coefficient $\bar{C}_{20}$, which is adequate for a “conventional tide free” model such as EGM96. When using a “zero tide” model, this permanent part should not be counted twice, this is the goal of Step 3 of the computation. See section 6.3.

With frequency-independent values $k_{nm}$ (Step 1), changes induced by the $(nm)$ part of the tide generating potential in the normalized geopotential coefficients having the same $(nm)$ are given in the time domain by

$$
\Delta \tilde{C}_{nm} - i \Delta \tilde{S}_{nm} = \frac{k_{nm}}{2n+1} \sum_{j=2}^{n} \frac{GM_j}{GM_\oplus} \left( \frac{R_e}{r_j} \right)^{n+1} \tilde{P}_{nm}(\sin \Phi_j) e^{-im\lambda_j} \quad (1)
$$

(with $\tilde{S}_{n0} = 0$), where

- $k_{nm}$ = nominal Love number for degree $n$ and order $m$,
- $R_e$ = equatorial radius of the Earth,
- $GM_\oplus$ = gravitational parameter for the Earth,
- $GM_j$ = gravitational parameter for the Moon ($j = 2$) and Sun ($j = 3$),
- $r_j$ = distance from geocenter to Moon or Sun,
- $\Phi_j$ = body fixed geocentric latitude of Moon or Sun,
- $\lambda_j$ = body fixed east longitude (from Greenwich) of Moon or Sun,

and $\tilde{P}_{nm}$ is the normalized associated Legendre function related to the classical (unnormalized) one by
\[ \bar{P}_{nm} = N_{nm} P_{nm}, \quad (2a) \]

where
\[ N_{nm} = \sqrt{\frac{(n-m)!(2n+1)(2-\delta_{om})}{(n+m)!}}. \quad (2b) \]

Correspondingly, the normalized geopotential coefficients \((\bar{C}_{nm}, \bar{S}_{nm})\) are related to the unnormalized coefficients \((C_{nm}, S_{nm})\) by
\[ C_{nm} = N_{nm} \bar{C}_{nm}, \quad S_{nm} = N_{nm} \bar{S}_{nm}. \quad (3) \]

Equation (1) yields \(\Delta \bar{C}_{nm}\) and \(\Delta \bar{S}_{nm}\) for both \(n = 2\) and \(n = 3\) for all \(m\), apart from the corrections for frequency dependence to be evaluated in Step 2. (The particular case \((nm) = (20)\) needs special consideration, however, as already indicated.)

One further computation to be done in Step 1 is that of the changes in the degree 4 coefficients produced by the degree 2 tides. They are given by
\[ \Delta \bar{C}_{4m} = \frac{k_{2m}^{(+)}}{3} \sum_{j=2}^{3} \frac{GM_{J}}{R_{J}^3} \bar{P}_{2m}(\sin \Phi_{j}) e^{-im\lambda_{j}}, \quad (4) \]

which has the same form as Equation (1) for \(n = 2\) except for the replacement of \(k_{2m}\) by \(k_{2m}^{(+)}\). The parameter values for the computations of Step 1 are given in Table 6.1. The choice of these nominal values has been made so as to minimize the number of terms for which corrections will have to be applied in Step 2. The nominal value for \(m = 0\) has to be chosen real because there is no closed expression for the contribution to \(\bar{C}_{20}\) from the imaginary part of \(k_{20}^{(0)}\).

<table>
<thead>
<tr>
<th>Table 6.1 Nominal values of solid Earth tide external potential Love numbers.</th>
<th>Elastic Earth</th>
<th>Anelastic Earth</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(n)</td>
<td>(m)</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.29525</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.29470</td>
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<td>0</td>
<td>0.093</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.093</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0.093</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.094</td>
</tr>
</tbody>
</table>

The frequency dependence corrections to the \(\Delta \bar{C}_{nm}\) and \(\Delta \bar{S}_{nm}\) values obtained from Step 1 are computed in Step 2 as the sum of contributions from a number of tidal constituents belonging to the respective bands. The contribution to \(\Delta \bar{C}_{20}\) from the long period tidal constituents of various frequencies \(f\) is
\[ Re \sum_{f(2,0)} (A_0 \delta k_f H_f) e^{i\theta_f} = \sum_{f(2,0)} [(A_0 H_f \delta k_f^R \cos \theta_f - (A_0 H_f \delta k_f^I) \sin \theta_f)], \quad (5a) \]
while the contribution to \((\Delta \bar{C}_{21} - i \Delta \bar{S}_{21})\) from the diurnal tidal constituents and to \(\Delta \bar{C}_{22} - i \Delta \bar{S}_{22}\) from the semidiurnals are given by
\[ \Delta \bar{C}_{2m} - i \Delta \bar{S}_{2m} = \eta_m \sum_{f(2,m)} (A_m \delta k_f H_f) e^{i\theta_f}, \quad (m = 1, 2), \quad (5b) \]
6.1 Effect of Solid Earth Tides

where

\[ A_0 = \frac{1}{R_e \sqrt{4\pi}} = 4.4228 \times 10^{-8} \text{ m}^{-1}, \]  
(5c)

\[ A_m = \frac{(-1)^m}{R_e \sqrt{8\pi}} = (-1)^m (3.1274 \times 10^{-8}) \text{ m}^{-1}, \quad (m \neq 0), \]  
(5d)

\[ \eta_1 = -i, \quad \eta_2 = 1, \]  
(5e)

\[ \delta k_f = \text{difference between } k_f \text{ defined as } k_{2m}^{(0)} \text{ at frequency } f \text{ and the nominal value } k_{2m}, \text{ in the sense } k_f - k_{2m}, \text{ plus a contribution from ocean loading}, \]

\[ \delta k_f^R = \text{real part of } \delta k_f, \text{ and} \]

\[ \delta k_f^I = \text{imaginary part of } \delta k_f, \text{ i.e., } \delta k_f = \delta k_f^R + i\delta k_f^I, \]

\[ H_f = \text{amplitude (in meters) of the term at frequency } f \text{ from the harmonic expansion of the tide generating potential, defined according to the convention of Cartwright and Tayler (1971), and} \]

\[ \theta_f = \bar{n} \cdot \bar{\beta} = \sum_{i=1}^{6} n_i \beta_i, \quad \text{or} \]

\[ \theta_f = m(\theta_g + \pi) - \bar{N} \cdot \bar{F} = m(\theta_g + \pi) - \sum_{j=1}^{5} N_j F_j, \]

where

\[ \bar{\beta} = \text{six-vector of Doodson's fundamental arguments } \beta_i, \]

\[ (\tau, s, h, p, N', p_s), \]

\[ \bar{n} = \text{six-vector of multipliers } n_i \text{ (for the term at frequency } f \text{) of the fundamental arguments}, \]

\[ \bar{F} = \text{five-vector of fundamental arguments } F_j \text{ (the Delaunay variables } l, l', F, D, \Omega) \text{ of nutation theory}, \]

\[ \bar{N} = \text{five-vector of multipliers } N_i \text{ of the Delaunay variables for the nutation of frequency } -f + d\theta_g/dt, \]

and \( \theta_g \) is the Greenwich Mean Sidereal Time expressed in angle units (i.e. \( 24^h = 360^o \); see Chapter 5).

(\( \pi \) in \((\theta_g + \pi)\) is now to be replaced by 180.)

For the fundamental arguments \((l, l', F, D, \Omega)\) of nutation theory and the convention followed here in choosing their multipliers \(N_j\), see Chapter 5.

For conversion of tidal amplitudes defined according to different conventions to the amplitude \(H_f\) corresponding to the Cartwright-Taylor convention, use Table 6.5 given at the end of this chapter.

For diurnal tides, the frequency dependent values of any load or body tide Love number parameter \(L\) (such as \(k_{21}^{(0)}\) or \(k_{31}^{(+)}\) in the present context) may be represented as a function of the tidal excitation frequency \(\sigma\) by a resonance formula

\[ L(\sigma) = L_0 + \sum_{\alpha=1}^{3} \frac{L_\alpha}{(\sigma - \sigma_\alpha)}, \]  
(6)

except for the small corrections referred to earlier. (They are to take account of frequency dependent contributions to a few of the Earth’s deformability parameters, which make (6) inexact.) The \(\sigma_\alpha\), \((\alpha = 1, 2, 3)\), are the respective resonance frequencies associated with the Chandler wobble (CW), the retrograde free core nutation (FCN), and the prograde free core nutation (PFCN, also known as the free inner core nutation, FICN), and the \(L_\alpha\) are the corresponding resonance coefficients.

All the parameters are complex. The \(\sigma_\alpha\) and \(\sigma\) are expressed in cycles per sidereal day, with the convention that positive (negative) frequencies represent retrograde (prograde) waves. (This sign convention, followed in tidal theory, is the opposite of that employed in analytical theories of
Table 6.2 Parameters in the resonance formulae for \( k_{21} \) and \( L_\alpha \) and the load Love numbers.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( k^{(0)} )</th>
<th>( k^{(+)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k^{(0)} )</td>
<td>( L_\alpha )</td>
<td>( L_\alpha )</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>( \text{Re} L_\alpha )</td>
<td>( \text{Im} L_\alpha )</td>
</tr>
<tr>
<td>0</td>
<td>0.29954</td>
<td>-0.1412 \times 10^{-2}</td>
</tr>
<tr>
<td>1</td>
<td>-0.77896 \times 10^{-3}</td>
<td>-0.3711 \times 10^{-4}</td>
</tr>
<tr>
<td>2</td>
<td>0.90963 \times 10^{-4}</td>
<td>-0.2963 \times 10^{-5}</td>
</tr>
<tr>
<td>3</td>
<td>-0.11416 \times 10^{-5}</td>
<td>0.5325 \times 10^{-7}</td>
</tr>
</tbody>
</table>

Load Love Numbers (Real parts only)

<table>
<thead>
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<th>( h_{21} )</th>
<th>( k_{21} )</th>
<th>( k_{21} )</th>
</tr>
</thead>
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<tr>
<td>0</td>
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<td>0.02315</td>
</tr>
<tr>
<td>1</td>
<td>1.6583 \times 10^{-3}</td>
<td>2.3232 \times 10^{-4}</td>
</tr>
<tr>
<td>2</td>
<td>2.8018 \times 10^{-4}</td>
<td>-8.4659 \times 10^{-6}</td>
</tr>
<tr>
<td>3</td>
<td>5.5852 \times 10^{-7}</td>
<td>1.0724 \times 10^{-8}</td>
</tr>
</tbody>
</table>

(\( n \) is the neutronium frequency in radians per second, and \( f \) is the tidal frequency in radians per second.) In particular, given the tidal frequency \( f \) in degrees per hour, one has

\[
\sigma = f/(15 \times 1.002737909),
\]

the factor 1.002737909 being the number of sidereal days per solar day. The values used herein for the \( \sigma_\alpha \) are from Mathews et al. (2002), adapted to the sign convention used here:

\[
\begin{align*}
\sigma_1 &= -0.0026010 - 0.0001361 i \\
\sigma_2 &= 1.0023181 + 0.000025 i \\
\sigma_3 &= 0.999026 + 0.000780 i.
\end{align*}
\]

They were estimated from a fit of nutation theory to precession rate and nutation amplitude estimates found from an analysis of very long baseline interferometry (VLBI) data.
The expressions given in the penultimate section of this chapter for the contributions from ocean tidal loading assume the constant nominal value \( k_2^{(\text{nom})} = -0.3075 \) for \( k' \) of the degree 2 tides. Further contributions arise from the frequency dependence of \( k_{21}^{'} \). These may be expressed, following Wahr and Sasao (1981), in terms of an effective ocean tide contribution \( \delta k_{21}^{(OT)}(\sigma) \) to the body tide Love number \( k_{21}^{(0)} \):

\[
\delta k_{21}^{(OT)}(\sigma) = [k_{21}^{'}(\sigma) - k_2^{(\text{nom})}](4\pi G \rho_w R/5g) A_{21}(\sigma), \tag{8}
\]

where \( G \) is the constant of universal gravitation, \( \rho_w \) is the density of sea water (1025 kg m\(^{-3}\)), \( R \) is the Earth’s mean radius (6.371 × 10\(^6\) m), \( g \) is the mean acceleration due to gravity at the Earth’s surface (9.820 m s\(^{-2}\)), and \( A_{21}(\sigma) \) is the admittance for the degree 2 tesseral component of the ocean tide of frequency \( \sigma \) cpsd:

\[
A_{21}(\sigma) = \zeta_{21}(\sigma)/\bar{H}(\sigma).
\]

\( \zeta_{21} \) is the complex amplitude of the height of the \((nm) = (21)\) component of the ocean tide, and \( \bar{H} \) is the height equivalent of the amplitude of the tide generating potential, the bar being a reminder that the spherical harmonics used in defining the two amplitudes should be identically normalized. Wahr and Sasao (1981) employed the factorized form

\[
A_{21}(\sigma) = f_{FCN}(\sigma) f_{OD}(\sigma),
\]

wherein the first factor represents the effect of the FCN resonance, and the second, that of other ocean dynamic factors. The following empirical formulae (Mathews et al., 2002) which provide good fits to the FCN factors of a set of 11 diurnal tides (Desai and Wahr, 1995) and to the admittances obtainable from the ocean load angular momenta of four principal tides (Chao et al., 1996) are used herein:

\[
\begin{align*}
  f_{OD}(\sigma) &= (1.3101 - 0.8098 \, i) - (1.1212 - 0.6030 \, i)\sigma, \\
  f_{FCN}(\sigma) &= 0.1732 + 0.9687 \, f_{eqm}(\sigma), \\
  f_{eqm}(\sigma) &= \frac{\gamma(\sigma)}{1 - (3\rho_w/5\bar{\rho})\gamma'(\sigma)},
\end{align*}
\]

where \( \gamma = 1 + k - h \) and \( \gamma' = 1 + k' - h' \), \( \bar{\rho} \) is the Earth’s mean density. (Here \( k \) stands for \( k_{21}^{(0)} \), and similarly for the other symbols. Only the real parts need be used.) \( f_{eqm} \) is the FCN factor for a global equilibrium ocean.

Table 6.3a shows the values of

\[
\delta k_f = (k_{21}^{(0)}(\sigma) - k_{21})(A_1 \delta k_f H_f),
\]

along with the real and imaginary parts of the amplitude \( (A_1 \delta k_f H_f) \). The tides listed are those for which either of the parts is at least \( 10^{-13} \) after round-off. (A cutoff at this level is used for the individual terms in order that accuracy at the level of \( 3 \times 10^{-12} \) be not affected by the accumulated contributions from the numerous smaller terms that are disregarded.) Roughly half the value of the imaginary part comes from the ocean tide term, and the real part contribution from this term is of about the same magnitude.
Table 6.3a The in-phase ($ip$) amplitudes ($A_{1}\delta k_{21}^{R}H_{f}$) and the out-of-phase ($op$) amplitudes ($A_{1}\delta k_{21}^{I}H_{f}$) of the corrections for frequency dependence of $k_{21}^{(0)}$, taking the nominal value $k_{21}$ for the diurnal tides as $(0.29830 - i 0.00144)$. Units: $10^{-12}$. The entries for $\delta k_{21}^{R}$ and $\delta k_{21}^{I}$ are in units of $10^{-5}$. Multipliers of the Doodson arguments identifying the tidal terms are given, as also those of the Delaunay variables characterizing the nutations produced by these terms.

<table>
<thead>
<tr>
<th>Name</th>
<th>deg/hr</th>
<th>Doodson</th>
<th>$\tau$</th>
<th>$s$</th>
<th>$h$</th>
<th>$p$</th>
<th>$N'$</th>
<th>$p_{s}$</th>
<th>$l$</th>
<th>$l'$</th>
<th>$F$</th>
<th>$D$</th>
<th>$\Omega$</th>
<th>$\delta k_{21}^{R}$</th>
<th>$\delta k_{21}^{I}$</th>
<th>Amp.</th>
<th>Amp.</th>
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<td>0</td>
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<td>0</td>
<td>2</td>
<td>-2</td>
<td>9</td>
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<tr>
<td>$\sigma_{1}$</td>
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<td>0</td>
<td>2</td>
<td>2</td>
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<td>0</td>
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</table>
The values used for $k^{(0)}_{21}(\sigma)$ in evaluating $\delta k_f$ are from an exact computation necessarily involving use of the framework of rotation-wobble theory which is outside the scope of this chapter. If the (approximate) resonance formula were used instead for the computation, the resulting numbers for $\delta k_f$ and $\delta k_f$ would require small corrections to match the exact values. In units of $10^{-5}$, they are (in-phase, out-of-phase) (1, 1) for $Q_1$, (1, 1) for $O_1$ and its companion having Doodson numbers 145,545, (1, 0) for $NO_1$, (0, 1) for $P_1$, (244, 299) for $\psi_1$, (12, 12) for $\phi_1$, (3, 2) for $J_1$, and (2, 1) for $OO_1$ and its companion with Doodson numbers 185,565. These are the only tides for which the corrections would contribute nonnegligibly to the numbers listed in the last two columns of the table.

Calculation of the correction due to any tidal constituent is illustrated by the following example for $K_1$. Given that $A_m = A_1 = -3.1274 \times 10^{-8}$, and that $H_f = 0.36870, \theta_f = (\theta_g + \pi)$, and $k^{(0)}_{21} = (0.25746+0.00118i)$ for this tide, one finds on subtracting the nominal value (0.29830–0.00144) that $\delta k_f = (-0.04084 + 0.00262i)$. Equation (5b) then yields:

$$\begin{align*}
(\Delta \bar{C}_{21})_{K_1} & = 470.9 \times 10^{-12} \sin(\theta_g + \pi) - 30.2 \times 10^{-12} \cos(\theta_g + \pi), \\
(\Delta \bar{S}_{21})_{K_1} & = 470.9 \times 10^{-12} \cos(\theta_g + \pi) + 30.2 \times 10^{-12} \sin(\theta_g + \pi). 
\end{align*}$$

The variation of $k^{(0)}_{21}$ across the zonal tidal band, $(nm) = (20)$, is due to mantle anelasticity; it is described by the formula

$$k^{(0)}_{21} = 0.29525 - 5.796 \times 10^{-4} \left\{ \cot \left( \frac{\sigma \pi}{2} \right) \left[ 1 - \left( \frac{f_m}{f} \right)^\alpha \right] + i \left( \frac{f_m}{f} \right)^\alpha \right\}$$

(9)

on the basis of the anelasticity model referred to earlier. Here $f$ is the frequency of the zonal tidal constituent, $f_m$ is the reference frequency equivalent to a period of 200 s, and $\alpha = 0.15$. The $\delta k_f$ in Table 6.3b are the differences between $k^{(0)}_{21}$ computed from the above formula and the nominal value $k_{21} = 0.30190$ given in Table 6.1.

The total variation in geopotential coefficient $\bar{C}_{21}$ is obtained by adding to the result of Step 1 the sum of the contributions from the tidal constituents listed in Table 6.3b computed using equation (5a). The tidal variations in $\bar{C}_{2m}$ and $\bar{S}_{2m}$ for the other $m$ are computed similarly, except that equation (5b) is to be used together with Table 6.3a for $m = 1$ and Table 6.3c for $m = 2$.

### 6.2 Solid Earth Pole Tide

The pole tide is generated by the centrifugal effect of polar motion, characterized by the potential

$$\Delta V(r, \theta, \lambda) = -\frac{\Omega r^2}{2} \sin 2\theta (m_1 \cos \lambda + m_2 \sin \lambda)$$

(10)

(See sub-section 7.1.4 for further details, including the relation of the wobble variables $(m_1, m_2)$ to the polar motion variables $(x_p, y_p)$.) The deformation which constitutes this tide produces a perturbation

$$-\frac{\Omega r^2}{2} \sin 2\theta \Re \left[ k_2 (m_1 - im_2) e^{i\lambda} \right]$$

in the external potential, which is equivalent to changes in the geopotential coefficients $\bar{C}_{21}$ and $\bar{S}_{21}$. Using for $k_2$ the value 0.3077 + 0.0036i appropriate to the polar tide yields

$$\begin{align*}
\Delta \bar{C}_{21} & = -1.333 \times 10^{-9} (m_1 - 0.0115m_2), \\
\Delta \bar{S}_{21} & = -1.333 \times 10^{-9} (m_2 + 0.0115m_1),
\end{align*}$$

where $m_1$ and $m_2$ are in seconds of arc.
Table 6.3b  Corrections for frequency dependence of $k_{20}^{(0)}$ of the zonal tides due to anelasticity. Units: $10^{-12}$. The nominal value $k_{20}$ for the zonal tides is taken as 0.30190. The real and imaginary parts $\delta k_f^R$ and $\delta k_f^I$ of $\delta k_f$ are listed, along with the corresponding in-phase ($ip$) amplitude ($A_0 H_f \delta k_f^R$) and out-of-phase ($op$) amplitude ($-A_0 H_f \delta k_f^I$) to be used in equation (5a).

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Table 6.3c  Amplitudes ($A_2 \delta k_f H_f$) of the corrections for frequency dependence of $k_{22}^{(0)}$, taking the nominal value $k_{22}$ for the sectorial tides as (0.30102 $- i \times 0.00130$). Units: $10^{-12}$. The corrections are only to the real part.

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6.3  Treatment of the Permanent Tide

The degree 2 zonal tide generating potential has a mean (time average) value that is nonzero. This time independent ($nm$) = (20) potential produces a permanent deformation and a consequent time independent contribution to the geopotential coefficient $\bar{C}_{20}$. In formulating a geopotential model, two approaches may be taken (see Chapter 1). When the time independent contribution is included in the adopted value of $\bar{C}_{20}$, then the value is termed “zero tide” and will be noted here $\bar{C}_{20}^{zt}$. This is the case for the JGM-3 model. If the time independent contribution is not included in the adopted value of $\bar{C}_{20}$, then the value is termed “conventional tide free” and will be noted here $\bar{C}_{20}^{tf}$. This is the case of the EGM96 model.

When using the EGM96 geopotential model as originally disseminated, i.e. as a “conventional tide free" model with $\bar{C}_{20}^{tf} = -0.4841653717 \times 10^{-3}$ at epoch 2000, the full tidal model given by (1), computed according to the preceding sections, should be used.
6.4 Effect of the Ocean Tides

In the case of a “zero tide” geopotential model, the model of tidal effects to be added should not once again contain a time independent part. One must not then use the expression (1) as it stands for modeling $\Delta C_{20}$; its permanent part must first be restored. This is Step 3 of the computation, which provides us with $\Delta \bar{C}_{20}$.

The symbol $\Delta \bar{C}_{20}$ will hereafter be reserved for the temporally varying part of (1) while the full expression will be redesignated as $\Delta C_{20}$ and the time independent part $\Delta C_{20}^{perm}$. Thus

$$\Delta \bar{C}_{20} = \Delta C_{20}^* - \langle \Delta C_{20}^* \rangle,$$  \hfill (11)

where

$$\Delta C_{20}^* = \frac{k_{20}}{5} \sum_{j=2}^{3} \frac{GM_j}{GM_{\oplus}} \left( \frac{R_0}{r_j} \right)^{3} P_{20}(\sin \Phi_j),$$

$$\langle \Delta C_{20}^* \rangle = A_0 H_0 k_{20} = (4.4228 \times 10^{-8})(-0.31460) k_{20}. \hfill (12)$$

In evaluating it, the same value must be used for $k_{20}$ in both $\Delta C_{20}^*$ and $\langle \Delta C_{20}^* \rangle$. With $k_{20} = 0.30190$, $\langle \Delta C_{20}^* \rangle = -4.201 \times 10^{-9}$. EGM96 has been computed using $k_{20} = 0.3$, therefore $\langle \Delta C_{20}^* \rangle = -4.173 \times 10^{-9}$ and $C_{20}^* = -0.484169382 \times 10^{-5}$ at epoch 2000.

The use of “zero tide” values and the subsequent removal of the effect of the permanent tide from the tide model is presented for consistency with the 18th IAG General Assembly Resolution 16.

6.4 Effect of the Ocean Tides

The dynamical effects of ocean tides are most easily incorporated by periodic variations in the normalized Stokes’ coefficients. These variations can be written as

$$\Delta \bar{C}_{nm} = i \Delta \bar{S}_{nm} = F_{nm} \sum_{s(n,m)} \sum_{+} \left( C^{\pm}_{nm} \pm i S^{\pm}_{nm} \right) e^{\pm i \theta_s}, \hfill (13)$$

where

$$F_{nm} = \frac{4 \pi G \rho_w}{g_c} \sqrt{\frac{(n+m)!}{(n-m)!(2n+1)\delta_{om}} \left( 1 + k'_n \right)},$$

$g_c$ and $G$ are given in Chapter 1, 

$\rho_w =$ density of seawater = 1025 kg m$^{-3}$, 

$k'_n =$ load deformation coefficients ($k'_2 = -0.3075$, $k'_3 = -0.195, k'_4 = -0.132, k'_5 = -0.1032, k'_6 = -0.0892$),

$C^{\pm}_{nm}, S^{\pm}_{nm}$ = ocean tide coefficients (m) for the tide constituent $s$

$\theta_s =$ argument of the tide constituent $s$ as defined in the solid tide model (Chapter 7).

The summation over $+$ and $-$ denotes the respective addition of the retrograde waves using the top sign and the prograde waves using the bottom sign. The $C^{\pm}_{nm}$ and $S^{\pm}_{nm}$ are the coefficients of a spherical harmonic decomposition of the ocean tide height for the ocean tide due to the constituent $s$ of the tide generating potential.

For each constituent $s$ in the diurnal and semidiurnal tidal bands, these coefficients were obtained from the CSR 3.0 ocean tide height model (Eanes and Bettadpur, 1995), which was estimated from the TOPEX/ Poseidon satellite altimeter data. For each constituent $s$ in the long period band, the self-consistent equilibrium tide model of Ray and Cartwright (1994) was used. The list of constituents for which the coefficients were determined was obtained from the Cartwright and Tayler (1971) expansion of the tide generating potential.
These ocean tide height harmonics are related to the Schwiderski convention (Schwiderski, 1983) according to
\[
C_{\pm snm}^s - iS_{\pm snm}^s = -i\hat{C}_{\pm snm}^s e^{i(\chi_{snm}^s + \epsilon_{snm}^s)},
\]
where
\[
\hat{C}_{\pm snm}^s = \text{ocean tide amplitude for constituent } s \text{ using the Schwiderski notation},
\]
\[
\epsilon_{snm}^s = \text{ocean tide phase for constituent } s, \text{ and}
\]
\[
\chi_{snm}^s = \text{is obtained from Table 6.4, with } H_s \text{ being the Cartwright and Tayler (1971) amplitude at frequency } s.
\]

Table 6.4 Values of \(\chi_s\) for long-period, diurnal and semidiurnal tides.

<table>
<thead>
<tr>
<th>Tidal Band</th>
<th>(H_s &gt; 0)</th>
<th>(H_s &lt; 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long Period</td>
<td>(\pi)</td>
<td>0</td>
</tr>
<tr>
<td>Diurnal</td>
<td>(\pi/2)</td>
<td>(-\pi/2)</td>
</tr>
<tr>
<td>Semidiurnal</td>
<td>0</td>
<td>(\pi)</td>
</tr>
</tbody>
</table>

For clarity, the terms in equation 1 are repeated in both conventions:
\[
\Delta \hat{C}_{nm} = F_{nm} \sum_{s(n,m)} [(C_{snm}^+ + C_{snm}^-) \cos \theta_s + (S_{snm}^+ + S_{snm}^-) \sin \theta_s] \quad (15a)
\]
or
\[
\Delta \hat{C}_{nm} = F_{nm} \sum_{s(n,m)} [\hat{C}_{snm}^+ \sin(\theta_s + \epsilon_{snm}^+ + \chi_s) + \hat{C}_{snm}^- \sin(\theta_s + \epsilon_{snm}^- + \chi_s)], \quad (15b)
\]
\[
\Delta S_{nm} = F_{nm} \sum_{s(n,m)} [(S_{snm}^+ - S_{snm}^-) \cos \theta_s - (C_{snm}^+ - C_{snm}^-) \sin \theta_s] \quad (15c)
\]
or
\[
\Delta S_{nm} = F_{nm} \sum_{s(n,m)} [\hat{C}_{snm}^+ \cos(\theta_s + \epsilon_{snm}^+ + \chi_s) - \hat{C}_{snm}^- \cos(\theta_s + \epsilon_{snm}^- + \chi_s)], \quad (15d)
\]

The orbit element perturbations due to ocean tides can be loosely grouped into two classes. The resonant perturbations arise from coefficients for which the order \((m)\) is equal to the first Doodson’s argument multiplier \(n_1\) of the tidal constituent \(s\) (See Note), and have periodicities from a few days to a few years. The non-resonant perturbations arise when the order \(m\) is not equal to index \(n_1\). The most important of these are due to ocean tide coefficients for which \(m = n_1 + 1\) and have periods of about 1 day.

Certain selected constituents (e.g. \(S_a\) and \(S_2\)) are strongly affected by atmospheric mass distribution (Chapman and Lindzen, 1970). The resonant harmonics (for \(m = n_1\)) for some of these constituents were determined by their combined effects on the orbits of several satellites. These multi-satellite values then replaced the corresponding values from the CSR 3.0 altimetric ocean tide height model.

Based on the predictions of the linear perturbation theory outlined in Casotto (1989), the relevant tidal constituents and spherical harmonics were selected for several geodetic and altimetric satellites. For geodetic tectonic satellites, both resonant and non-resonant perturbations were analyzed, whereas for altimetric satellites, only the non-resonant perturbations were analyzed. For the latter, the adjustment of empirical parameters during orbit determination removes the errors in modeling resonant accelerations. The resulting selection of ocean tidal harmonics was then...
merged into a single recommended ocean tide force model. With this selection the error of omission on TOPEX is approximately 5 mm along-track, and for Lageos it is 2 mm along-track. The recommended ocean tide harmonic selection is available via anonymous ftp from <ftp.csr.utexas.edu/pub/tide>.

For high altitude geodetic satellites like Lageos, in order to reduce the required computing time, it is recommended that out of the complete selection, only the constituents whose Cartwright and Tayler amplitudes $H_s$ is greater than 0.5 mm be used, with their spherical harmonic expansion terminated at maximum degree and order 8. The omission errors from this reduced selection on Lageos is estimated at approximately 1 cm in the transverse direction for short arcs.

NOTE: The Doodson variable multipliers ($\bar{n}$) are coded into the argument number (A) after Doodson (1921) as:

$$A = n_1(n_2 + 5)(n_3 + 5)(n_4 + 5)(n_5 + 5)(n_6 + 5).$$

6.5 Conversion of Tidal Amplitudes defined according to Different Conventions

The definition used for the amplitudes of tidal terms in the recent high-accuracy tables differ from each other and from Cartwright and Tayler (1971). Hartmann and Wenzel (1995) tabulate amplitudes in units of the potential (m$^2$s$^{-2}$), while the amplitudes of Roosbeek (1996), which follow the Doodson (1921) convention, are dimensionless. To convert them to the equivalent tide heights $H_f$ of the Cartwright-Tayler convention, multiply by the appropriate factors from Table 6.5. The following values are used for the constants appearing in the conversion factors: Doodson constant $D_1 = 2.63358352855$ m$^2$s$^{-2}$; $g_e \equiv g$ at the equatorial radius = 9.79828685 (from $GM = 3.986004415 \times 10^{14}$ m$^3$s$^{-2}$, $R_e = 6378136.55$ m).

Table 6.5 Factors for conversion to Cartwright-Tayler amplitudes from those defined according to Doodson’s and Hartmann and Wenzel’s conventions.

<table>
<thead>
<tr>
<th>From Doodson</th>
<th>From Hartmann &amp; Wenzel</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{20} = -\frac{\sqrt{10\pi}}{5\sqrt[3]{5}}\frac{D_1}{g_e}$</td>
<td>$f'_{20} = 2\frac{\sqrt{\pi}}{g_e} = 0.361788$</td>
</tr>
<tr>
<td>$f_{21} = -\frac{2\sqrt{2\pi}}{3\sqrt[3]{5}}\frac{D_1}{g_e}$</td>
<td>$f'_{21} = -\frac{\sqrt{2\pi}}{g_e} = -0.511646$</td>
</tr>
<tr>
<td>$f_{22} = \frac{\sqrt{10\pi}}{3\sqrt[3]{5}}\frac{D_1}{g_e}$</td>
<td>$f'_{22} = \frac{\sqrt{10\pi}}{g_e} = 0.511646$</td>
</tr>
<tr>
<td>$f_{30} = -\frac{\sqrt{10\pi}}{\sqrt[3]{7}}\frac{D_1}{g_e}$</td>
<td>$f'_{30} = \frac{\sqrt{10\pi}}{g_e} = 0.361788$</td>
</tr>
<tr>
<td>$f_{31} = \frac{\sqrt{10\pi}}{8\sqrt[3]{7}}\frac{D_1}{g_e}$</td>
<td>$f'_{31} = \frac{\sqrt{10\pi}}{g_e} = 0.511646$</td>
</tr>
<tr>
<td>$f_{32} = \frac{\sqrt{1440\pi}}{10\sqrt[3]{7}}\frac{D_1}{g_e}$</td>
<td>$f'_{32} = \frac{\sqrt{1440\pi}}{g_e} = 0.511646$</td>
</tr>
<tr>
<td>$f_{33} = -\frac{\sqrt{96\pi}}{15\sqrt[3]{7}}\frac{D_1}{g_e}$</td>
<td>$f'_{33} = \frac{\sqrt{96\pi}}{g_e} = -0.511646$</td>
</tr>
</tbody>
</table>

$^{11}$ftp.csr.utexas.edu/pub/tide
References


Eanes R. J. and Bettadpur, S., 1995, “The CSR 3.0 global ocean tide model,” Technical Memorandum CSR-TM-95-06, Center for Space Research, University of Texas, Austin, TX.


