WAVELET ANALYSIS OF STOCHASTIC SIGNALS

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1 INTRODUCTION

In the last years the wavelet transform has become a very appropriate tool for analysing geodetic observations. In opposite to the classical Fourier transform the most important feature of the wavelet transform is the detection of time-varying amplitudes and/or frequencies. The wavelet theory is described by many authors; see e.g. (CHUI 1994) or (LOUIS ET AL. 1998). Generally observed time series like the nutation values obtained by Very Long Baseline Interferometry (SCHMIDT AND SCHUH 1999) are stochastic signals. Thus, the theory of stochastic signals or processes has to be combined with wavelet theory. From this point of view some important foundations of the wavelet analysis of stochastic signals are presented in this paper. Especially two aspects, namely the variance distribution and the interpretation of the cross-covariance function, will be described in more detail and will be discussed in the last section. A numerical example of the procedure derived in this article is presented in the paper "Abilities of wavelet analysis for investigating short-period variations of Earth rotation" by SCHMIDT AND SCHUH (2000), also published in this volume.

2 FOURIER ANALYSIS

The most important component of Fourier analysis is the classical *Fourier transform*, defined by

$$X(j\omega) = \int_{\mathbb{R}} x(t)\, e^{-j\omega t}\, dt \quad \text{with} \quad j = \sqrt{-1}\,, \tag{2.1}$$

which transforms a continuous signal $x(t)$ with $x \in \mathbb{C}$ and $t \in \mathbb{R}$ from the time domain into the frequency domain represented by the angular frequency $\omega \in \mathbb{R}$. The Fourier transform may be modified by some kind of filtering and re-transformed into the time domain by the *inverse Fourier transform*

$$x(t) = \frac{1}{2\pi} \int_{\mathbb{R}} X(j\omega)\, e^{j\omega t}\, d\omega\,. \tag{2.2}$$

We assume that $x(t)$ is both element of the space $L^1(\mathbb{R})$ of the absolute integrable functions and element of the space $L^2(\mathbb{R})$ of the square integrable functions. Thus, the condition

$$\int_{\mathbb{R}} |x(t)|^2 \, dt = \int_{\mathbb{R}} x^\star(t) \, x(t) \, dt < \infty \tag{2.3}$$

holds, where $x^\star(t)$ denotes the conjugate complex function of $x(t)$. Applying Parseval's formula to equation (2.3) and substituting $x^\star(t) \, y(t)$ for $x^\star(t) \, x(t)$, we obtain the relation

$$\int_{\mathbb{R}} x^\star(t) \, y(t) \, dt = \frac{1}{2\pi} \int_{\mathbb{R}} X^\star(j\omega) \, Y(j\omega) \, d\omega \tag{2.4}$$

(PAPOULIS 1984, p.279). By analogy with (2.1) $Y(j\omega)$ denotes the Fourier transform of the signal $y(t)$.

In the next step we assume that the signal $x(t)$ represents a time series or *stochastic process.* Generally a *stationary* stochastic process is characterized by the feature that the statistical properties do not change with time (PRIESTLEY 1981, p.104). The process $x(t)$ is called *wide-sense stationary* if the *expected value function*

$$\mu_x(t) = E\big[x(t)\big] = \mu_x \tag{2.5}$$

is a constant and the *autocovariance function*

$$C_x(t_1, t_2) = E\big[(x^\star(t_1) - \mu_x^\star(t_1)) \, (x(t_2) - \mu_x(t_2))\big] = C_x(\tau) \tag{2.6}$$

depends only on the time difference $\tau = t_2 - t_1$. In the statistical usage the expected value function denotes the *first moment function* and the autocovariance function the *second central moment function* of the stochastic signal (KOCH AND SCHMIDT 1994, p.164). In the following the term "stationary" will mean "wide-sense stationary". Because of (2.3) and (2.5) the Fourier transform of a stationary stochastic process does not exist in general. Thus, according to the theorem of Wiener-Khintchine the frequency content is determined by the Fourier transform of the autocovariance function (2.6)

$$S_x(\omega) = \int_{\mathbb{R}} C_x(\tau) \, e^{-j\omega\tau} \, d\tau \; . \tag{2.7}$$

Since the relation $C_x(\tau) = C_x^\star(-\tau)$ holds the *Fourier spectrum* $S_x(\omega)$ is a real-valued function. Without loss of generality we assume that the expected value function (2.5) is equal to zero

$$\mu_x = 0 \, . \tag{2.8}$$

Frequently the autocovariance function (2.6) is not known. To derive a suitable estimation of the autocovariance function we suppose that the time series $x(t)$ is observed within a certain time intervall $T = t_2 - t_1$ with $t_1 \le t \le t_2$. Hence, the realisation

$$x_T(t) = \begin{cases} x(t) & \text{for} \quad t \in T \\ 0 & \text{for} \quad t \notin T \end{cases} \tag{2.9}$$

fulfills the condition (2.3) and the Fourier transform reads

$$X_T(j\omega) = \int_{\mathbb{R}} x_T(t) \, e^{-j\omega t} \, dt = \int_T x(t) \, e^{-j\omega t} \, dt \, . \tag{2.10}$$

If we suppose that the process $x_T(t)$ is *ergodic*, estimates of the expected value function and the autocovariance function are calculable from one single realisation (PRIESTLEY 1981, p.340).

Hence, a suitable but biased estimation of the autocovariance function $C_x(\tau)$ is given by the well-known formula

$$\tilde{C}_x(\tau) = \frac{1}{T} \int_{t_1}^{t_2-\tau} x^\star(t)\, x(t+\tau)\, dt \tag{2.11}$$

with $\tilde{C}_x(-\tau) = \tilde{C}_x^\star(\tau)$ for $0 \leq \tau \leq T$ (PRIESTLEY 1981, p.721). Substituting $\tilde{C}_x(\tau)$ for the autocovariance function $C_x(\tau)$ in equation (2.7) we obtain the estimation

$$\tilde{S}_x(\omega) = \int_{-T}^{T} \tilde{C}_x(\tau)\, e^{-j\omega\tau}\, d\tau \tag{2.12}$$

of the Fourier spectrum $S_x(\omega)$. Without going into details the Fourier spectrum may be calculated by means of the Fourier transform (2.10)

$$\tilde{S}_x(\omega) = \frac{1}{T} \left| X_T(j\omega) \right|^2 \tag{2.13}$$

(KOCH AND SCHMIDT 1994, p.220). According to (2.2) we compute the inverse Fourier transform of equation (2.12) and set $\tau = 0$. Thus, comparing this result with equation (2.11) we obtain the relations

$$E_{x_T} = T\, \tilde{\sigma}_x^2 = \int_T |x(t)|^2\, dt = \frac{T}{2\pi} \int_{\mathbb{R}} \tilde{S}_x(\omega)\, d\omega \tag{2.14}$$

for the total energy E_{x_T} of the stochastic process $x_T(t)$ defined in (2.9). The quantity $\tilde{\sigma}_x^2$ stands for the estimation of the *variance function* $\sigma_x^2 = C_x(0)$. It has to be mentioned that (2.14) is only valid if the assumption $\mu_x = 0$ is applied to the expected value function. Otherwise we have to substitute the second moment function of the signal for the second central moment function (KOCH UND SCHMIDT 1994, p.164). However, we recognize from equation (2.14) that with $\mu_x = 0$ the estimated variance $\tilde{\sigma}_x^2$ is a measure of the average power of the stochastic signal $x_T(t)$.

Analogously to (2.9) we introduce the realisation $y_T(t)$ of a stationary and ergodic process $y(t)$ with the expected value function $\mu_y = 0$ and the autocovariance function $C_y(\tau) = E\big[y^\star(t)\, y(t+\tau)\big]$. According to (2.12) and (2.13) an estimation $\tilde{S}_{x,y}(j\omega)$ of the *Fourier cross-spectrum* $S_{x,y}(j\omega)$ is given by

$$\tilde{S}_{x,y}(j\omega) = \int_{-T}^{T} \tilde{C}_{x,y}(\tau)\, e^{-j\omega\tau}\, d\tau = \frac{1}{T}\, X_T^\star(j\omega)\, Y_T(j\omega)\,, \tag{2.15}$$

where the function

$$\tilde{C}_{x,y}(\tau) = \frac{1}{T} \int_{t_1}^{t_2-\tau} x^\star(t)\, y(t+\tau)\, dt \tag{2.16}$$

with $\tilde{C}_{x,y}(-\tau) = \tilde{C}_{y,x}^\star(\tau)$ and $0 \leq \tau \leq T$ is a biased estimation of the *cross-covariance function* $C_{x,y}(\tau)$. According to (2.10) $Y_T(j\omega)$ denotes the Fourier transform of the process $y_T(t)$. Generally the estimation $\tilde{S}_{x,y}(j\omega)$ is a complex-valued function with $\tilde{S}_{x,y}(j\omega) = \tilde{S}_{y,x}^\star(j\omega)$. The function

$$\Gamma_{x,y}(j\omega) = \frac{S_{x,y}(j\omega)}{\big(S_x(\omega)\, S_y(\omega)\big)^{1/2}} \tag{2.17}$$

with $0 \leq |\Gamma_{x,y}(j\omega)| \leq 1$ is a measure for the similarity of the two processes $x_T(t)$ and $y_T(t)$ in dependence on the angular frequency ω. Whereas the so-called *coherency* $\Gamma_{x,y}(j\omega)$ may be interpreted as a correlation function in the frequency domain, the *cross-correlation function*

$$\rho_{x,y}(\tau) = \frac{C_{x,y}(\tau)}{\sigma_x\, \sigma_y} \tag{2.18}$$

with $0 \leq |\rho_{x,y}(\tau)| \leq 1$ describes the correlation structure between the two processes $x_T(t)$ and $y_T(t)$ in the time domain. Estimates for the variances $\sigma_x^2 = C_x(0)$ and $\sigma_y^2 = C_y(0)$ are computed from equation (2.11) setting $\tau = 0$ as mentioned above. The square of equation (2.17) is called *normed coherency* (MEIER AND KELLER 1990, p.68).

3 WAVELET ANALYSIS

The main disadvantage of the classical Fourier transform (2.1) is the fact that it determines the frequency content of the signal $x(t)$ without any information about the location of the frequencies in the time domain. The *continuous wavelet transform*

$$X(b,a) = \int_{\mathbb{R}} x(t)\, \psi_{a,b}^{\star}(t)\, dt \tag{3.1}$$

analyses the signal $x(t)$ in the two-dimensional *phase space* spanned by the *shift parameter* $b \in \mathbb{R}$ and the *scale parameter* $a \in \mathbb{R}\backslash\{0\}$ (CHUI 1992, p.5). Thus, the functions

$$\psi_{a,b}(t) = |a|^{-1/2}\, \psi\Big(\frac{t-b}{a}\Big)\,, \tag{3.2}$$

which are generally localized both in the time domain and the frequency domain, are substituted for the complex oscillations $e^{j\omega t}$ of infinite length of the Fourier transform (2.1). The Fourier transform

$$\Psi(j\omega) = \int_{\mathbb{R}} \psi(t)\, e^{-j\omega t}\, dt \tag{3.3}$$

of the *wavelet function* $\psi(t)$ satisfies the so-called *admissibility condition*

$$C_\psi = \int_{\mathbb{R}} \frac{|\Psi(j\omega)|^2}{|\omega|}\, d\omega < \infty \tag{3.4}$$

which may be rewritten into the equivalent form

$$\int_{\mathbb{R}} \psi(t)\, dt = 0\,. \tag{3.5}$$

The Fourier transform of the function $\psi_{a,b}(t)$ from equation (3.2) is given by

$$\Psi_{a,b}(j\omega) = |a|^{1/2}\, \Psi(ja\omega)\, e^{-j\omega b}\,. \tag{3.6}$$

Thus, applying Parseval's formula (2.4) to equation (3.1) we obtain the wavelet transform with respect to the frequency domain

$$X(b,a) = \frac{|a|^{1/2}}{2\pi} \int_{\mathbb{R}} X(j\omega)\, \Psi^{\star}(ja\omega)\, e^{j\omega b}\, d\omega\,, \tag{3.7}$$

if the Fourier transform $X(j\omega)$ of the signal $x(t)$ exists. An example for a wavelet function often used in geodetic and geophysical applications is the *Morlet function*; see e.g. (SCHMIDT AND SCHUH 1999) or (SCHMIDT AND SCHUH 2000).

The main advantage of the wavelet transform is given by the fact that the signal is analysed in an adaptive time-frequency window. Thus, the width of the window is narrow for the high-frequency components and wide for the low-frequency parts of the signal. However, the width and the height of the window are restricted to the uncertainty principle, which means that the

area of the resolution cell remains independent of the location in the phase space.

Again we assume that the signal $x(t)$ is stationary and ergodic. Since the condition (3.5) holds for the wavelet function $\psi(t)$ the expected value function $E\big[X(b,a)\big] = \mu_X$ of the wavelet transform (3.1) of a stationary signal is always equal to zero. To derive the autocovariance function $E\big[X^\star(b_1,a)\,X(b_2,a)\big] = C_X(b_1,b_2,a)$ of the wavelet transform we first evaluate the product $X^\star(b_1,a)\,X(b_2,a)$. After inserting the wavelet transform (3.1) once for $b = b_1$ and once again for $b = b_2$ and considering equation (3.2) the expression

$$\begin{aligned} X^\star(b_1,a)\,X(b_2,a) &= |a|^{-1}\Big(\int_{\mathbb{R}} x^\star(t_1)\,\psi\big(\frac{t_1-b_1}{a}\big)\,dt_1\Big)\Big(\int_{\mathbb{R}} x(t_2)\,\psi^\star\big(\frac{t_2-b_2}{a}\big)\,dt_2\Big) \\ &= |a|^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}} \psi\big(\frac{t_1-b_1}{a}\big)\,\psi^\star\big(\frac{t_2-b_2}{a}\big)\,x^\star(t_1)\,x(t_2)\,dt_1\,dt_2 \end{aligned}$$

follows. Setting $t_2 = t_1 + \tau$ we recognize that the autocovariance function

$$C_X(b_1,b_2,a) = \int_{\mathbb{R}} C_x(\tau)\int_{\mathbb{R}} \psi^\star_{a,s}(\tau)\,\psi_{-a,\beta}(s)\,ds\,d\tau = C_X(\beta,a) \tag{3.8}$$

of the wavelet transform only depends on the scale parameter a and the difference $\beta = b_2 - b_1$ between the two values b_1 and b_2 for the shift parameter b. Hence, the wavelet transform (3.1) of a stationary signal is stationary, too. Now we define the family of so-called *zeta-functions*

$$\zeta_{a,\beta}(\tau) = \int_{\mathbb{R}} \psi_{a,s}(\tau)\,\psi^\star_{-a,\beta}(s)\,ds = \int_{\mathbb{R}} \psi_{a,\beta}(t)\,\psi^\star_{a,\tau}(t)\,dt = \zeta\big(\frac{\tau-\beta}{a}\big) \tag{3.9}$$

and rewrite equation (3.8)

$$C_X(\beta,a) = \int_{\mathbb{R}} C_x(\tau)\,\zeta^\star_{a,\beta}(\tau)\,d\tau\,. \tag{3.10}$$

Since the function $\zeta(\tau)$ from (3.9) satisfies the condition (3.5), we conclude from equation (3.10) that the autocovariance function $C_X(\beta,a)$ of the wavelet transform (3.1) results from a wavelet transform of the autocovariance function $C_x(\tau)$ by means of the wavelet function $\zeta(\tau)$. Applying Parseval's formula (2.4) to equation (3.10) we finally obtain

$$C_X(\beta,a) = \frac{|a|}{2\pi}\int_{\mathbb{R}} S_x(\omega)\,|\Psi(ja\omega)|^2\,e^{j\omega\beta}\,d\omega\,. \tag{3.11}$$

Hence, in the frequency domain the autocovariance function $C_X(\beta,a)$ is calculated from the integration of the weighted Fourier spectrum $S_x(\omega)$. The weight function $|a|\;|\Psi(ja\omega)|^2$ only depends on the scale parameter a.

After substituting the estimation (2.13) for the Fourier spectrum $S_x(\omega)$ into equation (3.11) the expression

$$\tilde{C}_X(\beta,a) = \frac{1}{2\pi T}\int_{\mathbb{R}} \Big(|a|^{1/2}\,X^\star_T(j\omega)\,\Psi(ja\omega)\Big)\Big(|a|^{1/2}\,X_T(j\omega)\,\Psi^\star(ja\omega)\,e^{j\omega\beta}\Big)\,d\omega \tag{3.12}$$

follows. Applying Parseval's formula (2.4) again and considering equation (3.7) the estimated autocovariance function reads

$$\tilde{C}_X(\beta,a) = \frac{1}{T}\int_{\mathbb{R}} X^\star_T(b,a)\,X_T(b+\beta,a)\,db\,, \tag{3.13}$$

where

$$X_T(b,a) = \int_{\mathbb{R}} x_T(t)\,\psi^\star_{a,b}(t)\,dt = \int_T x(t)\,\psi^\star_{a,b}(t)\,dt \tag{3.14}$$

denotes the wavelet transform (3.1) of the stochastic signal $x_T(t)$ defined in (2.9). A suitable alternative to the computation of the autocovariance function by means of equation (3.13) is given by equation (3.10) after inserting the estimation (2.11) of the autocovariance function $C_x(\tau)$

$$\tilde{C}_X(\beta, a) = \int_{-T}^{T} \tilde{C}_x(\tau)\ \zeta^{\star}_{a,\beta}(\tau)\ d\tau\ . \tag{3.15}$$

Setting $\beta = 0$ in the equations (3.11) and (3.13) we obtain the relations

$$\tilde{\sigma}^2_X(a) = \frac{|a|}{2\pi} \int_{\mathbb{R}} \tilde{S}_x(\omega)\ |\Psi(ja\omega)|^2\ d\omega = \frac{1}{T} \int_{\mathbb{R}} |X_T(b,a)|^2\ db\ , \tag{3.16}$$

for the estimation $\tilde{\sigma}^2_X(a)$ of the scale-dependent *variance function* $\sigma^2_X(a) = C_X(0,a)$ of the wavelet transform (3.14). From the integration of equation (3.16) over the scale domain we derive the expressions

$$E_{x_T} = \frac{T}{C_\psi} \int_{\mathbb{R}} \tilde{\sigma}^2_X(a)\ \frac{da}{a^2} = \frac{1}{C_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}} |X_T(b,a)|^2\ db\ \frac{da}{a^2} \tag{3.17}$$

for the total energy E_{x_T} of the signal $x_T(t)$ considering the admissibility condition (3.4) and the definition (2.14) (KUMAR AND FOUFOULA-GEORGIOU 1994, p.13). Thus, the *wavelet scalogram*

$$W_x(b,a) = |X_T(b,a)|^2 \tag{3.18}$$

may be interpreted as the *energy density function* depending on the shift parameter b and the scale parameter a. Equating (2.14) and (3.17) we finally obtain the relation

$$\tilde{\sigma}^2_x = \frac{1}{C_\psi} \int_{\mathbb{R}} \tilde{\sigma}^2_X(a)\ \frac{da}{a^2} \tag{3.19}$$

between the estimated values of the total variance σ^2_x of the signal and the scale-dependent variances $\sigma^2_X(a)$ of the wavelet transform.

Again we introduce the realisation $y_T(t)$ of the stochastic signal $y(t)$ and define corresponding to the equations (3.13) and (3.15) the estimation

$$\tilde{C}_{X,Y}(\beta, a) = \int_{-T}^{T} \tilde{C}_{x,y}(\tau)\ \zeta^{\star}_{a,\beta}(\tau)\ d\tau = \frac{1}{T} \int_{\mathbb{R}} X^{\star}_T(b,a)\ Y_T(b+\beta, a)\ db \tag{3.20}$$

of the *cross-covariance function* $C_{X,Y}(\beta, a)$. According to (3.14) $Y_T(b,a)$ denotes the wavelet transform of the process $y_T(t)$. The estimation

$$\tilde{\sigma}_{X,Y}(a) = \frac{1}{T} \int_{\mathbb{R}} W_{x,y}(b,a)\ db \tag{3.21}$$

of the scale-dependent wavelet *covariance function* $\sigma_{X,Y}(a) = C_{X,Y}(0,a)$ is computed from (3.20) setting $\beta = 0$. By analogy with (3.18) the complex-valued function

$$W_{x,y}(b,a) = X^{\star}_T(b,a)\ Y_T(b,a) \tag{3.22}$$

is called the *wavelet cross-scalogram*. Finally we define the *cross-correlation function*

$$\rho_{X,Y}(\beta, a) = \frac{C_{X,Y}(\beta, a)}{\sigma_X(a)\ \sigma_Y(a)} \tag{3.23}$$

with $0 \leq |\rho_{X,Y}(\beta,a)| \leq 1$ depending on the difference $\beta = b_2 - b_1$ and the scale parameter a. Estimates for the cross-covariance function $C_{X,Y}(\beta,a)$ and the variances $\sigma^2_X(a)$ and $\sigma^2_Y(a)$ are calculated from the equations (3.20) and (3.16), respectively. For $\beta = 0$ we call the function $|\Gamma_{X,Y}(a)|^2$ with $\rho_{X,Y}(0,a) = \Gamma_{X,Y}(a)$ the *normed coherency* of the wavelet transform.

4 CONCLUSIONS

The main objective of this article is to combine the theory of stochastic signals or processes with the wavelet theory. Generally, a stochastic signal or time series is characterized by its moment functions, namely the expected value function and the autocovariance function. Based on the autocovariance function of a stationary and ergodic signal and the cross-covariance function of two stochastic signals of the same kind, the following two aspects are the most important results derived in the previous section:

- According to equation (3.19) the estimated value $\tilde{\sigma}_x^2$ of the total variance σ_x^2 of the stationary stochastic signal $x_T(t)$, defined in (2.9) is decomposed into the scale-dependent variance components $\tilde{\sigma}_X^2(a)$, defined in equation (3.16). As mentioned in the context of equation (2.14) the value of the total variance is equal to the average power of a signal with zero mean. Thus, the dominant structures of the signal are detected by means of the wavelet scalogram (3.18), which represents the energy density function. Because of the fact that the angular frequency ω is related to the scale parameter a of the wavelet transform, the wavelet analysis determines time-dependent frequencies of stochastic signals.

- The cross-covariance function $C_{x,y}(\tau)$ determines the time lag between similar structures of two signals $x(t)$ and $y(t)$. By means of the wavelet transform (3.10) a scale-dependent time lag function is derived. Since the scale parameter a is related to the angular frequency ω, the described procedure allows the introduction of frequency-dependent time lags in the excitation functions for modelling polar motion and length of day variations.

An example for the application of the procedure derived in this article is presented in the paper "Abilities of wavelet analysis for investigating short-period variations of Earth rotation" by SCHMIDT AND SCHUH (2000, this volume).

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